An application of polycyclic monoids to rings

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Communicated by N. Reilly

Any set of elements $\{e_{i,j} : 1 \leq i, j \leq n\}$ in a ring R which satisfy

$$e_{i,j}e_{p,q} = \delta_{j,p}e_{i,q} \text{ and } \sum_{i=1}^{n} e_{i,i} = 1$$

is called a *set of* $n \times n$*-matrix units* in R. It can be proved (Proposition 3.18 of [2]) that a ring R contains a set of $n \times n$-matrix units if, and only if, R is isomorphic to a full matrix ring $M_n(S)$ for some ring S. This result can also be interpretted in terms of embeddings of inverse semigroups. The finite, combinatorial Brandt semigroup B_n [7] is isomorphic to the set of $n \times n$-matrix units. The existence of a set of matrix units in a ring R is equivalent to an embedding (preserving zeros) of B_n into the multiplicative semigroup of R such that the sum of the images of the idempotents of B_n is the identity of the ring R. We call such an embedding *strong.* The results of this paper can be regarded as a refinement of this result.

An important technique in the K-theory of certain C^*-algebras is that of 'halving projections'. The existence of such projections in a C^*-algebra A leads to an isomorphism between A and the ring of all $n \times n$-matrices over A [8]. This technique is also employed by Girard in his work on linear logic [3]. In this note, we show that such a process is equivalent to the existence of a 'strong embedding' of the polycyclic monoid on two generators. We also provide a semigroup theoretic analogue of this result. We shall assume that the reader is familiar with the elementary theory of inverse semigroups [7].

The *polycyclic inverse monoid on* α *generators*, P_α, where α is countable, is the inverse monoid generated by the set $\{p_i : 1 \leq i \leq \alpha\}$, subject to the relations $p_i p_j^{-1} = \delta_{ij}$. It will be convenient to denote the generators of P_2 by $p = p_1$ and $q = p_2$. The polycyclic monoids were introduced by Nivat and Perrot [6], and rediscovered by Cuntz within the theory of C^*-algebras [1]. The polycyclic monoid on one generator is referred to as the *bicyclic monoid.* Embeddings of the bicyclic monoid into rings were considered by Jacobson [5] (though not in these terms), and we shall not consider them further here (we would like to thank Chris Robson for this reference). Thus throughout the remainder of this paper $\alpha \geq 2$.

The presence of polycyclic monoids in a particular context usually indicates that there are self-similarity phenomena present. For example, P_2 can be represented by partial injective maps on the Cantor set: the elements p^{-1} and q^{-1} are mapped to the self-embeddings that map the whole Cantor set to its left and right hand sides respectively. The main results of this paper will be further illustrations of this idea.

Our first result shows that the polycyclic monoid on two generators contains copies of all finitely generated polycyclic monoids.

Lemma 1. *There is an embedding of P_n into P_2 for each finite $n \geq 2$.*

Proof. We construct an inverse subsemigroup of P_2 isomorphic to P_n. Define a subset $\{p_{n,i} : 1 \leq i \leq n\}$ of P_2 as follows: $p_{n,i} = pq^{i-1}$ for $i = 1, \ldots, n-1$ and $p_{n,n} = q^{n-1}$. It is easy to check that $p_{n,i}p_{n,i}^{-1} = 1$ and $p_{n,i}p_{n,j}^{-1} = 0$ if $i \neq j$, using $pp^{-1} = 1 = qq^{-1}$ and $pq^{-1} = 0 = qp^{-1}$ respectively. The set $\{p_{n,i} : 1 \leq i \leq n\}$ generates an inverse submonoid P_n' of P_2 which is a homomorphic image of P_n. However, for $n \geq 2$ the monoid P_n is congruence free [6]. Thus P_n is isomorphic to P_n'. ■

Let R be a ring with identity. We say that P_n is *embedded in R* if there is a monoid homomorphism (preserving zeros) from P_n to the multiplicative monoid of R. In what follows we shall always identity P_n with its image to simplify notation. We say that P_n is *strongly embedded* in R if it satisfies the additional condition

$$\sum_{i=1}^{n} p_i^{-1} p_i = 1.$$

Proposition 2. *Let R be a ring. Then*
(i) *If P_n is embedded in R, then $e = \sum_{i=1}^{n} p_i^{-1} p_i$ is an idempotent, and eRe is isomorphic to $M_n(R)$.*
(ii) *If P_n is strongly embedded in R, then R is isomorphic to $M_n(R)$.*

Proof. (i) The element e is an idempotent since it is a sum of pairwise orthogonal idempotents. Define

$$\mathbf{h}_n = \begin{pmatrix} p_1^{-1} \\ p_2^{-1} \\ \cdots \\ p_n^{-1} \end{pmatrix} \text{ and } \mathbf{v}_n = \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{pmatrix}$$

Observe that $\mathbf{v}_n \mathbf{h}_n^t = I_n$, the $n \times n$-identity matrix, and $\mathbf{h}_n^t \mathbf{v}_n = e$.

Define $\Phi_n : M_n(R) \to eRe$ by $\Phi_n(A) = \mathbf{h}_n^t A \mathbf{v}_n$. This is well-defined for

$$e\Phi_n(A)e = (\mathbf{h}_n^t \mathbf{v}_n)(\mathbf{h}_n^t A \mathbf{v}_n)(\mathbf{h}_n^t \mathbf{v}_n) = \mathbf{h}_n^t A \mathbf{v}_n = \Phi_n(A).$$

It is also a ring homomorphism since for any matrices $A, B \in M_n(R)$ we have that

$$\Phi_n(A)\Phi_n(B) = (\mathbf{h}_n^t A \mathbf{v}_n)(\mathbf{h}_n^t B \mathbf{v}_n) = \mathbf{h}_n^t A (\mathbf{v}_n \mathbf{h}_n^t) B \mathbf{v}_n = \mathbf{h}_n^t A I_n B \mathbf{v}_n$$

which is just $\Phi_n(AB)$. It is clear that addition is preserved, and that the identity I_n of $M_n(R)$ is mapped to e, the identity of eRe.

Define $\Psi_n : eRe \to M_n(R)$ by $\Psi_n(r) = \mathbf{v}_n r \mathbf{h}_n^t$. This is a ring homomorphism since for any elements $r, s \in eRe$ we have that

$$\Psi_n(r)\Psi_n(s) = (\mathbf{v}_n r \mathbf{h}_n^t)(\mathbf{v}_n s \mathbf{h}_n^t) = \mathbf{v}_n (r(\mathbf{h}_n^t \mathbf{v}_n)s) \mathbf{h}_n^t = \mathbf{v}_n (res) \mathbf{h}_n^t = \mathbf{v}_n rs \mathbf{h}_n^t$$

which is just $\Psi_n(rs)$. It is clear that addition is preserved and that e is mapped to I_n.

Finally, for all $r \in eRe$,

$$\Phi_n(\Psi_n(r)) = \mathbf{h}_n^t \mathbf{v}_n r \mathbf{h}_n^t \mathbf{v}_n = ere = r,$$

and for all $A \in M_n(R)$,

$$\Psi_n(\Phi_n(A)) = \mathbf{v}_n \mathbf{h}_n^t A \mathbf{v}_n \mathbf{h}_n^t = I_n A I_n = A.$$

Hence, Φ and Ψ are mutually inverse ring isomorphisms.
(ii) This is immediate from (i), since in this case e = 1. ■

Proposition 3. *If P_2 is strongly embedded in R, then so is P_n, for all finite $n \geq 2$.*

Proof. We use Lemma 1. The result holds trivially for $n = 2$. Now observe that, with the notation from the proof of Lemma 1,

$$\sum_{j=1}^{n+1} p_{n+1,j}^{-1} p_{n+1,j} = p^{-1}p + q^{-1}\left(\sum_{i=1}^{n} p_{n,i}^{-1} p_{n,i}\right) q.$$

It follows that if P_n is strongly embedded in R, then P_{n+1} is too. Hence the result follows by induction. ■

The following is now immediate from Propositions 2 and 3.

Theorem 4. *If P_2 is strongly embedded in a ring R, then $M_n(R)$ is isomorphic to R for all finite n.* ■

An example of a ring satisfying the condition of Theorem 4 is the ring $B(l^2)$, of all bounded linear operators on infinite square-summable sequences. This property of $B(l^2)$ was used by J.-Y. Girard in [3].

The nearest analogue of matrix rings for semigroups are the Rees matrix semigroups. If S is a semigroup and $a \in S$ then a binary operation $\cdot$ may be defined on S by $s \cdot t = sat$. The set S is a semigroup with respect to this operation called a *variant* of S [4]. We denote the semigroup $(S, \cdot)$ by (S, a). In the results which follow, the symbol I will either denote the set $\{1, \ldots, n\}$ or the set $\mathbb{N}\backslash\{0\}$.

Proposition 5. *Let S be a monoid with zero which contains a copy of P_α, where α is countable. Let $\mathcal{M} = \mathcal{M}^0(S; I, I; Q)$ be a square Rees matrix semigroup over S with card$(I) = \alpha$ and $Q = [q_{ij}]$. If there is an element $q \in S$ such that $p_i q p_j^{-1} = q_{i,j}$ for $1 \leq i, j \leq \alpha$ then $\mathcal{M}$ can be embedded in the variant (S, q).*

Proof. Denote the zero of $\mathcal{M}$ by $\mathbf{0}$, where we identify each $(i, 0, j)$ with $\mathbf{0}$. Define a function $\theta : \mathcal{M} \to S$ by $\theta(i, s, j) = p_i^{-1} s p_j$. We show that θ is a homomorphism. Let (i, s, j) and (k, t, l) be any two elements of $\mathcal{M}$. Now $(i, s, j)(k, t, l) = (i, s q_{j,k} t, l)$. By definition

$$\theta(i, s q_{j,k} t, l) = p_i^{-1}(s q_{j,k} t) p_l$$

and

$$\theta(i, s, j) \cdot \theta(k, t, l) = (p_i^{-1} s p_j) q (p_k^{-1} t p_l);$$

but by assumption $p_j q p_k^{-1} = q_{j,k}$. Finally, to show that θ is injective suppose that $\theta(i, s, j) = \theta(k, t, l)$. Then $p_i^{-1} s p_j = p_k^{-1} t p_l$. Thus $s = p_i p_k^{-1} t p_l p_j^{-1}$ and $t = p_k p_i^{-1} s p_j p_l^{-1}$. If either $i \neq k$ or $j \neq l$ then $s = t = 0$ and so $(i, s, j) = (k, t, l) = \mathbf{0}$. On the other hand, if $i = k$ and $j = l$, then $s = t$, and again $(i, s, j) = (k, t, l)$. ■

Theorem 6. *If $\mathcal{M} = \mathcal{M}^0(I(\mathbb{N}); I, I; Q)$ is a square Rees matrix semigroup over the symmetric inverse monoid $I(\mathbb{N})$ in which I is countable and Q is diagonal, then $\mathcal{M}$ may be embedded in a variant of $I(\mathbb{N})$.*

Proof. Let α be the cardinality of I. Observe that $I(\mathbb{N})$ contains a copy of P_α: if α is finite let p_i^{-1} be the bijection from $\mathbb{N}$ to $\alpha\mathbb{N} + (i-1)$; if α is infinite take any bijection θ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ and let $p_{i+1}^{-1}(x) = \theta(i,x)$. Define q to be the union of the disjoint partial injections $p_i^{-1} q_{i,i} p_i$. It is easy to check that q satisfies the condition of Proposition 5. Thus $\mathcal{M}$ is embedded in $(I(\mathbb{N}), q)$. ■

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