Under consideration for publication in Math. Struct. in Comp. Science

Dynamical Algebras as self-similar objects in Unique Decomposition Categories

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Received 17 April 2009

The term "dynamical algebra" is used for a variety of monoids with partial summation featuring in the Geometry of Interaction program and related work. We demonstrate that these monoids arise from the interaction of two themes of the Geometry of Interaction program: Unique Decomposition Categories, which allow for matrix decompositions and categorical traces, and Self-Similarity, which gives the connectives, and – implicitly – an essentially untyped system. We demonstrate that all the algebraic structures referred to as "the dynamical algebra" are endomorphism monoids of self-similar objects in Unique Decomposition Categories, and this motivates the abstract definition.

UDCs allow for categorical analogues of finite-dimensional linear algebra, and dynamical algebras have similar structures in the monoid setting, including the countably infinite case. There is also a natural notion of 'bases' for dynamical algebras, as embeddings of an interesting class of inverse semigroups (the Polycyclic Monoids). We give a theory of changes of basis, diagonalisations, matrix representations 'up to associativity', and other analogues of linear algebra.

1. Introduction

It would be no exageration to claim that the *dynamical algebra* is the algebraic core of the Geometry of Interaction program (J.-Y. Girard 1988(i); J.-Y. Girard 1988(ii)), and related logical models (V. Danos, L. Regnier 1993; V. Danos, L. Regnier 1994). We demonstrate that this algebraic structure is consequence of the Unique Decomposition Category structure (used to model cut-elimination (E. Haghverdi, P. Scott 2004)) and reflexivity (required for a type-free structure, as in the combinatory algebra of (S. Abramsky, E. Haghverdi, P. Scott 2000) or the C–monoids of (J. Lambek, P. Scott 1986)).

In what follows, we identify dynamical algebras as structures arising from self-similarity of an object in a unique decomposition category. Unique Decomposition Categories provide categorical analogues of finite-dimensional linear algebra (including finite matrix decomposition of arrows between objects, and categorical traces based on iteration), the assumption of self-similarity allows for analogues of arbitrary linear algebraic structures within an (enriched) monoid.

2. Definitions — The dynamical algebra, inverse semigroups, polycyclic monoids

There are a number of possibly inequivalent definitions of 'the dynamical algebra' — the definitions of (J.-Y. Girard 1988(i)), (V. Danos, L. Regnier 1993), (J.-Y. Girard 1994) are not immediately equivalent. We first give the purely algebraic definition of (V. Danos, L. Regnier 1993).

In this definition, the dynamical algebra is a *semiring* — defined analogously to a ring, but without inverses; we refer to (J. Howie (1995)) for a fuller definition. We also refer to (C. Faith (1981)) for the basic theory, including the *free semiring* over a monoid.

Definition 1. *Semigroup rings, the free semiring*
Given a semigroup S, and commutative unital ring R, the **semigroup ring** R_S is defined in (J. Howie (1995)):
The elements of R_S are functions $\alpha : S \to R$, where $\alpha(s)$ is non-zero for at most a finite number of elements of S. Addition is the usual addition of maps into an abelian group, and composition is given by $(\alpha\beta)(s) = \sum_{uv=s} \alpha(u)\beta(v)$.

There are canonical embeddings of both S and R into R_S, as demonstrated in (S. Lang 1993). It is usual to quotient R_S by the ideal identifying all sums of the zero of R — we assume that this quotient has been applied, and refer to (C. Faith (1981)) for the algebraic details. In the case when the semiring is the natural numbers $\mathbb{N}$, we refer to this as the **free semiring on** S.

Definition 2. *Generalised inverses, inverse semigroups, inverse monoids*
An **inverse semigroup** is defined to be a semigroup S (that is, a set with an associative binary operation) where every element $s \in S$ has a unique **generalised inverse** s^{-1} satisfying $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. This does *not* imply that $ss^{-1} = 1 = s^{-1}s$; inverse semigroups are not even required to have a muliplicative identity. When an inverse semigroup M has a multiplicative identity 1, it is called an **inverse monoid**; however, the condition $ss^{-1} = 1 = s^{-1}s$ is still *not* part of the definition.

We refer to (M. V. Lawson 1998) for the general theory of inverse semigroups, including their definitions as relations on generators, and an exposition of the following example:

Definition 3. *Polycyclic monoids*
The n^{th} polycyclic monoid P_n is defined in (M. Nivat, J.-F. Perrot 1970) to be the inverse semigroup given by an n element generating set $\{p_0, p_1, \ldots, p_{n-1}\}$, together with the relations $p_i p_j^{-1} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Of particular importance is the polycyclic monoid on 2 generators, P_2, where it is customary to denote the generating set $\{p, q\}$, and give the relations explicitly as

$$pp^{-1} = 1 = qq^{-1} \quad , \quad pq^{-1} = 0 = qp^{-1}$$

All polycyclic monoids P_k have a zero, except for P_1. This inverse semigroup has very different properties to the other polycylic monoids, and is usually referred to as the **bicyclic monoid**, (M. Nivat, J.-F. Perrot 1970) — it plays no further part in this paper.

Theorem 1.
(i) Words in the polycyclic monoid P_n have normal form $v^{-1}w$, for $v, w \in \{p_0, \dots, p_{n-1}\}^*$, and composition of words in normal form is given by

$$(v^{-1}w)(x^{-1}y) = \begin{cases} v^{-1}hy & \text{when } w = hx \text{ for some } h \in \{p_0, \dots, p_{n-1}\}^* \\ v^{-1}k^{-1}y & \text{when } x = kw \text{ for some } k \in \{p_0, \dots, p_{n-1}\}^* \end{cases}$$

(ii) All polycyclic monoids, apart from the bicyclic monoid, are *congruence-free*; that is, any composition-preserving equivalence relation on P_n is either the universal congruence $r \sim s$ for all $r, s \in P_n$, or the identity congruence $r \sim s \;\Leftrightarrow\; r = s$ for all $r, s \in P_n$

Proof. We refer to (M. Nivat, J.-F. Perrot 1970) for these results. We also observe that in the context of rewriting systems, congruence-freeness is an example of Hilbert-Post completeness. □

Definition 4. *The dynamical algebra of (V. Danos, L. Regnier 1993)*
The **dynamical algebra** as presented in (V. Danos, L. Regnier 1993) is the free semiring on the polycyclic monoid P_2, subject to the *additive* quotient $p^{-1}p + q^{-1}q = 1$.

Discussion
(i) The full definition of the dynamical algebra given in (V. Danos, L. Regnier 1993) has additional structure — a unary operation denoted !( ), used to represent the *exponentials* of linear logic. We do not consider this separately, but derive such operations in Section 9.
(ii) Although the definition of the dynamical algebra given in (V. Danos, L. Regnier 1993) allows for addition of arbitrary finite sets of elements, the application given (modelling the untyped λ-calculus) only sums elements satisfying certain very strong conditions; we consider this from Proposition 3 onwards.

2.1. *Representations of inverse semigroups, and other definitions of the dynamical algebra*

The theory of inverse semigroups is the theory of partial symmetries, in the same way that group theory is the theory of symmetries. In group theory, this is formalised by the representation theorem as permutation groups; similarly, the representation theorem for inverse semigroups, given by (G. B. Preston 1954; V. V. Wagner 1952), gives inverse semigroups as partial injections on sets.

Notation Although $I(X)$ is standard semigroup-theoretic notation for the inverse monoid of all partial injections on a set, we use the category-theoretic notation of $\mathbf{pInj}(X, X)$ — the endomorphism monoid of partial invertible maps at an object X.

2.2. *Representations of polycyclic monoids*

A number of distinct representations of polycyclic monoids have been given. We present two distinct representations, one based on a countable set, and the other based on an uncountable set. For details of these, along with (a large number of) other representations, we refer to (P. Hines 1997; M. V. Lawson 1998).

Proposition 2.
(i) Let $\mathbb{N}$ denote the natural numbers. The submonoid of $\mathbf{pInj}(\mathbb{N},\mathbb{N})$ generated by the maps $p^{-1}(n) = 2n$ and $q^{-1}(n) = 2n+1$, together with their partial inverses

$$p(n) = \begin{cases} \frac{n}{2} & \text{n is even,} \\ \text{undefined} & \text{otherwise} \end{cases} \quad , \quad q(n) = \begin{cases} \frac{n-1}{2} & \text{n is odd,} \\ \text{undefined} & \text{otherwise} \end{cases}$$

is isomorphic to P_2.
(ii) Let $\mathcal{C} = \{0,1\}^\infty$ be the Cantor set, as all one-sided infinite words over $\{0,1\}$. The partial bijective maps $p^{-1}(\omega) = 0.\omega$ and $q^{-1}(\omega) = 1.\omega$, together with their partial inverses

$$p(\omega) = \begin{cases} \omega' & \text{when } \omega = 0.\omega' \\ \text{undefined} & \text{otherwise} \end{cases} \quad , \quad q(\omega) = \begin{cases} \omega' & \text{when } \omega = 1.\omega' \\ \text{undefined} & \text{otherwise} \end{cases}$$

generate an inverse submonoid of $\mathbf{pInj}(\mathcal{C},\mathcal{C})$ isomorphic to P_2:

Proof. In both cases, it is immediate that the conditions for the generators of the polycyclic monoid P_2 are satisfied. Hence, as they are clearly distinct, our result follows by the congruence-freeness given in Theorem 1. □

The connection between these representations and the dynamical algebra is that representations of inverse semigroups have a very natural, albeit partial, notion of addition:

Proposition 3. Let M be an inverse submonoid of $\mathbf{pInj}(X,X)$. Then the set-theoretic union $\bigcup_{i\in I} m_i : X \to X$ of some family $\{m_i : X \to X\}_{i\in I}$ is also a partial injection, provided the condition $m_i^{-1}m_j = 0 = m_j m_i^{-1}$ for all $i \neq j \in I$ is satisfied.

Proof. The condition $m_i^{-1}m_j = 0$ states that m_i and m_j have disjoint domains, and the condition $m_j m_i^{-1} = 0$ states that m_i and m_j have disjoint images. The set-theoretic union of a family of partial injections with pairwise-disjoint disjoint domains / images is also a partial injection, and so our result follows. □

Observation Note that all the sums used in the model of untyped λ-calculus presented in (V. Danos, L. Regnier 1993) satisfy this condition.

3. Two other definitions of the dynamical algebra

The dynamical algebra is presented in (J.-Y. Girard 1988(i)) in two distinct ways, both as representations:

Definition 5. *The dynamical algebra according to (J.-Y. Girard 1988(i))*
(i) The dynamical algebra is defined in (J.-Y. Girard 1988(i)) as the representation of P_2 in $\mathbf{pInj}(\mathbb{N},\mathbb{N})$ given in Proposition 2, together with the partial addition provided by set-theoretic union (Proposition 3).
(ii) The second definition gives the dynamical algebra as the representation of P_2 in $\mathbf{pInj}(\mathcal{C},\mathcal{C})$ given in Proposition 2, again, with partial addition provided by set-theoretic

union.

Discussion The definition (**i**) above was presented in (J.-Y. Girard 1988(i)) in terms of the action of these representations on a labelled basis set for a countably infinite Hilbert space (thus providing the Cuntz algebra of (J. Cuntz 1977)). However, it has often been observed that this is just a representation of the natural numbers operations — precisely, it is in the image of **pInj** under Barr's $l_2 : \mathbf{pInj} \rightarrow \mathbf{Hilb}$ functor (M. Barr 1992).

We now present a purely categorical derivation of the dynamical algebra that gives a definition broad enough to account for the three representations presented so far.

4. The dynamical algebra, arising from category theory

We present a method of constructing dynamical algebras (i.e. a family of monoids with a partial summation, satisfying what we claim are the required conditions). This is in terms of the theory of Unique Decomposition Categories (E. Haghverdi 2000), combined with the categorical theory of self-similarity (P. Hines 1999). We first present these theories separately, along with some motivation for the definitions, and then demonstrate how dynamical algebras arise naturally.

We assume some familiarity with category theory and refer to (S. MacLane 1971) for an introduction, including the theory of symmetric monoidal categories.

4.1. *Unique decomposition categories*

Unique decomposition categories were introduced in (E. Haghverdi 2000) in order to give an abstract setting for notions of *iteration* and 'particle-style' *categorical traces* (see (S. Abramsky 1996) for the intuition of 'categorical trace as iteration') used in a number of systems based on the Geometry of Interaction, including the original Geometry of Interaction series (J.-Y. Girard 1988(i); J.-Y. Girard 1988(ii)). UDCs are categories enriched over Σ - monoids (E. Manes, M. Arbib 1986), with additional conditions on a monoidal tensor. An interesting special case is that of partially additive monoids. We again refer to (E. Manes, M. Arbib 1986) for an exposition of partially additive categories, but present the remaining definitions below.

Definition 6. Σ-*monoids, Unique Decomposition Categories*
A **Sigma monoid** is a set M, together with a partial 'summation' operation $\Sigma : P(M) \rightarrow M$ defined on countable subsets of M, that satisfies the following axioms

1 (The unary-sum axiom) Every singleton subset of M is summable, and $\sum\{m\} = m$ for all $m \in M$.

2 (The Partition-Associativity Axiom) If $\{m_i\}_{i\in I} \subseteq M$ is a countable subset of M, and $\{I_j\}_{j\in J}$ is a partition of I, then $\sum_{i\in I} m_i$ exists exactly when $\sum_{i'\in I_j} m_{i'}$ exists and

$\sum_{j\in J}\left(\sum_{i'\in I_j} m_{i'}\right)$ exists. In this case,

$$\sum_{i\in I} m_i \;=\; \sum_{j\in J}\left(\sum_{i'\in I_j} m_{i'}\right)$$

A **Unique Decomposition Category**, abbreviated **UDC**, is defined to be a symmetric monoidal category $(\mathbf{C},\oplus)$ enriched over Σ-monoids, satisfying the following condition:

— For any object $\bigoplus_{i=1}^n X_i$, there exist **quasi-projections** and **quasi-inclusions**

$$\pi_k : \bigoplus_{i=1}^n X_i \to X_k \ , \ \iota_k : X_k \to \bigoplus_{i=1}^n X_i$$

satisfying $\pi_i\iota_j = \begin{cases} 1 & i=j \\ 0 & \text{otherwise}\end{cases}$ and $\sum_{i=1}^n \iota_i\pi_i = \ 1_{\oplus_{i=1}^n X_i}$

Note that, despite the $\oplus$ notation for the monoidal tensor, it is *not* required to be a coproduct. Also, these conditions do *not* imply that there exists a unique decomposition of any *object* X into some $\bigoplus_{i=1}^k X_i$. Rather, the 'unique decomposition' is of arrows between objects given in this form.

Proposition 4. Matrix representations of arrows
Let $X = \oplus_{i=1}^n X_i$ and $Y = \oplus_{j=1}^m Y_j$ be objects in a unique decomposition category. Then any arrow $f : X \to Y$ has a decomposition as $\{f_{ij} : X_j \to Y_i\}_{i=1..n,j=1.m}$, where f_{ij} is given by $f_{ij} = \pi_i f \iota_j : X_j \to Y_i$.

This decomposition may be written in matrix form as

$$f = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2m} \\ \dots & & & \dots \\ f_{m1} & f_{m2} & \dots & f_{mn} \end{pmatrix}$$

and composition of arrows in this form is given by the familiar formula for matrix multiplication $(gf)_{ki} = \sum_{j=1}^m g_{kj} f_{ji}$.

Proof. we refer to (E. Haghverdi 2000) for this result, along with questions of existence and uniqueness. □

4.2. *Self-similar objects and categorical self-similarity*

The categorical theory of self-similarity, based on a self-similar object $N \cong N \otimes N$, is given in (P. Hines 1999), where it is used to construct functors from categories to monoids. These functors preserve a number of categorical structures, such as a symmetric monoidal structure, categorical trace, compact closure, projections and inclusions, &c. (and may even impose coherence conditions on a category via the Karoubi envelope construction, as in (P. Hines 2003)), and so can be considered as the passage from typed to untyped systems.

Definition 7. *Self-similar objects*
Let $(\mathbf{C}, \otimes)$ denote a monoidal category. An object $N \in Ob(\mathbf{C})$ is called a **self-similar object** when $N \cong N \otimes N$ – that is, there exist isomorphisms $c_{N\otimes N} : N \otimes N \to N$ and $d_{N\otimes N} : N \to N \otimes N$ satisfying $dc = 1_{N\otimes N}$ and $cd = 1_N$.

Provided $(\mathbf{C}, \otimes)$ is associative up to isomorphism (we refer to (P. Hines 2003) for the strict associativity case), the maps $d : N \to N \otimes N$ and $c : N \otimes N \to N$ give rise to a monoidal functor from $\mathbf{N}^{\otimes}$ to the endomorphism monoid of N, considered as a one-object category. This is demonstrated in (P. Hines 1999); however, we reprise the basic definitions.

Definition 8. *Monogenic categories, compression and division isomorphisms*
Let N be a self-similar object in a monoidal category $(\mathbf{C}, \otimes)$. For all objects X in the **monogenic category** $N^{\otimes}$ (that is, the full subcategory of $\mathbf{C}$ generated by N and $\otimes$) we define the **compression** and **division isomorphisms** $c_X : X \to N$ and $d_X : N \to X$ inductively by:

— $c_{U\otimes V} = c_{N\otimes N}(c_U \otimes c_V)$.
— $d_{U\otimes V} = (d_U \otimes d_V)d_{N\otimes N}$.

It may be verified that $c_X d_X = 1_N$ and $d_X c_X = 1_X$, and this identity gives rise to a monoidal functor from the monogenic category $N^{\otimes}$ to the monoid $\mathbf{C}(N, N)$:

Lemma 5. Given a self-similar object N in a monoidal category $(\mathbf{C}, \otimes)$, then :
(**(i)** There exists a monoid homomorphism $\otimes_\phi : End(N) \times End(N) \to End(N)$, that satisfies an untyped analogue of MacLane's pentagon condition.
(**(ii)** When $(\mathbf{C}, \otimes)$ is symmetric, $\otimes_\phi$ also satisfies an untyped analogue of the commutativity hexagon.

Proof. The monoid homomorphism is defined by $u \otimes_\phi v = c(u \otimes v)d$, for all $u, v : N \to N$. We refer to (P. Hines 1999) for the remainder of this proof, including the analogues of MacLane's pentagon and hexagon conditions. ☐

Definition 9. *Unitless Monoidal Categories*
Note that neither $\mathbf{N}^{\otimes}$ nor $(\mathbf{End(N)}, \otimes_\phi)$, are monoidal categories - the definition of a monoidal category calls for unit objects, and corresponding units isomorphisms. However, the assumption that N is the unit object I of $\mathbf{C}$ gives $End(N)$ as an abelian monoid. We are interested in the case where $N \neq I$, so $End(N)$ has non-trivial structure. The somewhat awkward terminology **unitless monoidal categories** is used in (P. Hines 1999) for categories that have all the required structure for a monoidal category, apart from units.

For conciseness, we abuse notation and refer to monoidal tensors, and monoidal functors, even when there are no units objects or arrows. In the case where the presence or otherwise of units is important, we mention these explicitly.

Theorem 6. Given a self-similar object N in a monoidal category $(\mathbf{C},\otimes)$, there exists a monoidal functor from the category $\mathbf{N}^{\otimes}$ to the monoid $\mathbf{C}(N,N)$ (treated as a one-object category, with monoidal tensor given by $\otimes_\phi$).

Proof. We define the functor $\phi : \mathbf{N}^{\otimes} \rightarrow \mathbf{End}(\mathbf{N})$ by

— On objects, $\phi(X) = N$, for all $X \in Ob(\mathbf{N}^{\otimes})$.
— On arrows, $\phi(f) = c_Y f d_X$, for all $f : X \rightarrow Y$.

It may readily be verified from the definitions of c_Y, d_X and $\otimes_\phi$ that

— $\phi(1_X) = 1_N$ for all X
— $\phi(gf) = \phi(g)\phi(f)$ when this is defined
— $\phi(f \otimes f') = \phi(f) \otimes_\phi \phi(f')$

□

Definition 10. *The self-embedding functor, internalisations of tensor products*
We refer to the monoidal functor ϕ of Theorem 6 as the **self-embedding functor**, and the monoid tensor analogue $\otimes_\phi$ of Lemma 5 as the **internalisation of** $\otimes$ **by** ϕ, or just the **internalised tensor** when the context is unambiguous.

We also emphasise that no claims about *uniqueness* are made; from Definition 12 onwards, the non-uniqueness of internal tensors is an integral part of the theory.

4.3. *Combining self-similarity and unique decomposition categories*

We now consider endomorphism monoids of a self-similar objects in unique decomposition categories. Our claim is that they have exactly the structure required by the various definitions of the dynamical algebra.

Lemma 7. Let N be a self-similar object in a unique decomposition category $(\mathbf{C},\oplus)$. Then the self-embedding functor $\phi : \mathbf{N}^{\oplus} \rightarrow \mathbf{End}(\mathbf{N})$ preserves addition: for arbitrary summable family of arrows $\{F_i : X \rightarrow Y\}_{i\in I}$,

$$\phi\left(\sum_{i\in I} F_i\right) = \sum_{i\in I} \phi(F_i)$$

Proof. This follows from the distributivity of composition over summation, and the definition of the self-embedding functor; observe that

$$\phi\left(\sum_{i\in I} F_i\right) = c_Y \left(\sum_{i\in I} F_i\right) d_X = \sum_{i\in I} c_Y F_i d_X = \sum_{i\in I} \phi(F_i)$$

□

Theorem 8. Let N be a self-similar object in a UDC $(\mathbf{C},\oplus)$. Then there exists an injective monoid homomorphism $\eta : P_2 \rightarrow End(N)$ of the polycyclic monoid on two generators into $End(N)$, and this embedding is **strong** in that $\eta(p^{-1}p) + \eta(q^{-1}q) = 1$.

Proof. Consider the object $N \oplus N$ in the monogenic category $\mathbf{N}^{\oplus} \hookrightarrow \mathbf{C}$, together with the projection maps $\pi_1, \pi_2 : N \oplus N \rightarrow N$ and the inclusion maps $\iota_1, \iota_2 : N \rightarrow N \oplus N$

required by Definition 6. As the self-embedding map ϕ is a functor

$$\phi(\pi_1)\phi(\iota_1) = 1_N = \phi(\pi_2)\phi(\iota_2)$$

Similarly,

$$\phi(\iota_1)\phi(\pi_1) = 1 \oplus_\phi 0 \quad , \quad \phi(\iota_2)\phi(\pi_2) = 0 \oplus_\phi 1$$

Since $\oplus_\phi$ is a monoid homomorphism, $(1 \oplus_\phi 0)(0 \oplus_\phi 1) = 0 \oplus_\phi 0 = 0$, and so

$$\phi(\iota_1)\phi(\pi_1)\phi(\iota_2)\phi(\pi_2) = 0 = \phi(\iota_2)\phi(\pi_2)\phi(\iota_1)\phi(\pi_1).$$

Therefore, $\phi(\pi_1)\phi(\iota_2) = 0 = \phi(\pi_2)\phi(\iota_1)$. The embedding of P_2 into $End(N)$ then follows by the map (on generators)

$$\eta(p) = \phi(\pi_1) \qquad \eta(q) = \phi(\pi_2)$$

$$\eta(p^{-1}) = \phi(\iota_1) \quad \eta(q^{-1}) = \phi(\iota_2)$$

As P_2 is congruence-free η is either an injection or trivial, and we may observe that (for example) $\phi(\pi_1) \neq \phi(\pi_2)$. Therefore these 4 elements exactly generate an embedding of P_2 into $End(N)$. Finally, this embedding is strong, since from Lemma 7,

$$p^{-1}p+q^{-1}q = \phi(\iota_1)\phi(\pi_1)+\phi(\iota_2)\phi(\pi_2) = \phi(\iota_1\pi_1)+\phi(\iota_2\pi_2) = \phi(\iota_1\pi_1+\iota_2\pi_2) = \phi(1_N) = 1$$

□

Observation The natural numbers is a self-similar object in the category $(\mathbf{pInj}, \sqcup)$ with self-embedding given by the Cantor map $(n,i) \mapsto 2n+i$ for $(n,i) \in \mathbb{N} \times \{0,1\} \cong \mathbb{N} \sqcup \mathbb{N}$ and the Cantor set is a self-similar object in the category of partial homeomorphisms on topological spaces, being homeomorphic to both its left and right hand sides. This makes the link between the above abstract definition and the representations given in Proposition 5.

The link with the purely algebraic definition given in Definition 4 follows from the Wagner-Preston representation theorem for inverse semigroups (G. B. Preston 1954; V. V. Wagner 1952), and the observation that the only sums used in the λ-calculus models of (V. Danos, L. Regnier 1993) are those summable in the category of partial injections.

5. A categorical definition of dynamical algebras, and matrix representations

We have shown that endomorphism monoids of self-similar objects in UDCs have both embeddings of P_2, and a partial summation. This gives enough structure to model the connectives (– excluding exponentials) of linear logic as described in (J.-Y. Girard 1988(i)), and the related λ-calculus model (V. Danos, L. Regnier 1993). Because of this, we make a definition of dynamical algebras in these terms.

Definition 11. *Dynamical algebras, weak dynamical algebras, weakly self-similar objects* We define a **dynamical algebra** to be an endomorphism monoid of a self-similar object in a UDC. We refer to the various examples in the literature as *'the dynamical algebra of __'*, to avoid confusion.

Some presentations of the dynamical algebra *do not* assume that $p^{-1}p+q^{-1}q=1$. We call these **weak dynamical algebras**. This is equivalent to the condition that instead of having a *self-similar* object $N\cong N\otimes N$, we have a **weakly self-similar object** C that satisfies $C\otimes C\hookrightarrow C$ — that is, there exists arrows

$$s:C\rightarrow C\otimes C \quad , \quad r:C\otimes C\rightarrow C$$

satisfying

$$sr=1_{C\otimes C} \quad , \quad rs\neq 1_C$$

We refer to (P. Hines 2003) for analogues of the self-embedding functor, but do not consider weak self-similarity for the remainder of this paper.

Theorem 9.
Let N be a self-similar object of a Unique Decomposition Category $(\mathbf{C},\oplus)$, and let $c:N\oplus N\rightarrow N$ and $d:N\rightarrow N\oplus N$ be fixed isomorphisms generating a self-embedding functor $\phi:\mathbf{N}^{\oplus}\rightarrow\mathbf{End}(\mathbf{N})$. Then:
(i) Given a matrix decomposition of an arrow $F:N\oplus N\rightarrow N\oplus N$ as

$$F=\left(\begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22}\end{array}\right),$$

the image of F under the self-embedding functor $\phi:\mathbf{N}^{\oplus}\rightarrow\mathbf{End}(\mathbf{N})$ is given by

$$\phi(F)=p^{-1}F_{11}p+p^{-1}F_{12}q+q^{-1}F_{21}p+q^{-1}F_{22}q$$

(ii) The internalised tensor of the endomorphism monoid $\mathbf{End}(\mathbf{N})$ may be written as, for all $f,g:N\rightarrow N$,

$$f\oplus_\phi g=p^{-1}fp+q^{-1}gq:N\rightarrow N$$

(iii) A dynamical algebra is $\mathcal{D}$ is isomorphic to the algebra of 2×2 matrices over $\mathcal{D}$, with the isomorphism given by

$$d\mapsto\left(\begin{array}{cc} pdp^{-1} & pdq^{-1} \\ qdp^{-1} & qdq^{-1}\end{array}\right)$$

Proof.
(i) By Proposition 4 an arrow $F:N\oplus N\rightarrow N\oplus N$ has a matrix representation as $F=\left(\begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22}\end{array}\right)$, where $F_{ij}=\pi_iF\iota_j$ for $i,j\in\{1,2\}$, and so $F=\sum_{i,j\in\{1,2\}}\iota_jF\pi_i$. Applying the self-embedding functor gives

$$\phi(F)=p^{-1}F_{11}p+p^{-1}F_{12}q+q^{-1}F_{21}p+q^{-1}F_{22}q$$

where

$$\begin{array}{rr} p=\phi(\pi_1) & q=\phi(\pi_2) \\ p^{-1}=\phi(\iota_1) & q^{-1}=\phi(\iota_2)\end{array}$$

(ii) This is immediate from part **(i)** above, and the matrix representation of the monoidal tensor of $\mathbf{C}$ as $R\oplus S=\left(\begin{array}{cc} R & 0 \\ 0 & S\end{array}\right)$.

(iii) This is a simple corollory of the above, together with the matrix decomposition allowed in unique decomposition categories. ☐

Observation The above 'internalised tensor', written in terms of an embedding of the polycyclic monoid P_2, is used as the representation of the multiplicative linear logic connectives in the Geometry of Interaction series (J.-Y. Girard 1988(i); J.-Y. Girard 1988(ii)).

6. Non-uniqueness of self-embeddings as 'changes of basis'

In the examples given of self-similar objects in unique decomposition categories, the isomorphisms demonstrating the self-similarity (equivalently, the strong embeddings of the polycyclic monoid P_2) are not unique. For example, consider the odd-even representation of P_2 as partial bijections on the natural numbers (Proposition 2). Clearly, any partition of $\mathbb{N}$ into two distinct infinite subsets may also be used to generate an embedding of P_2, and there is an immediate 1:1 correspondence between such partitions and the elements of the Cantor set! Hence, although we have so far used a fixed self-embedding isomorphism (giving a single internalised tensor), in reality there are uncountably many in this example.

Observe also that the (2×2) matrix representation of a dynamical algebra, as presented in Theorem 9(iii), is dependent on the self-embedding chosen. Because of this, we make the following definitions.

Definition 12. *Bases of dynamical algebras*
Let $\mathcal{D} = End(N)$ be a dynamical algebra in a UDC $(\mathbf{C}, \oplus)$. We define a **basis** for $\mathcal{D}$ to be either:
(i) A strong embedding of P_2 into $\mathcal{D}$.
(ii) A pair of isomorphisms $c : N \oplus N \to N$, $d : N \oplus N \to N$.
(iii) A self-embedding functor $\phi : \mathbf{N}^{\oplus} \to \mathbf{End}(\mathbf{N})$.

Lemma 10. The three distinct definitions given in Definition 12 above are equivalent.

Proof. We have already seen that (**ii**) $\Rightarrow$ (**iii**) in (P. Hines 1999), and (**iii**) $\Rightarrow$ (**i**) in Theorem 8. It remains to show that (**i**) $\Rightarrow$ (**ii**).

Consider a dynamical algebra $\mathcal{D} = End(N)$ in some category $\mathbf{C}, \oplus$, together with a fixed strong embedding $\phi : P_2 \to \mathcal{D}$. Using the matrix decomposition allowed in UDCs, we may define $c : N \oplus N \to N$ and $d : N \to N \oplus N$ by

$$d = \begin{pmatrix} \phi(p) \\ \phi(q) \end{pmatrix} \quad , \quad c = \begin{pmatrix} \phi(p^{-1}) & \phi(q^{-1}) \end{pmatrix}$$

Observe that

$$dc = \begin{pmatrix} \phi(p) \\ \phi(q) \end{pmatrix} \begin{pmatrix} \phi(p^{-1}) & \phi(q^{-1}) \end{pmatrix} = \begin{pmatrix} \phi(pp^{-1}) & 0 \\ 0 & \phi(qq^{-1}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly,

$$cd = \left(\begin{array}{cc} \phi(p^{-1}) & \phi(q^{-1}) \end{array} \right) \left(\begin{array}{c} \phi(p) \\ \phi(q) \end{array} \right) = \phi(p^{-1}p) + \phi(q^{-1}p) = 1$$

Our result then follows immediately. □

Given that the above three definitions of a basis for a dynamic algebra are equivalent, we may define *changes of basis*:

Definition 13. *Changes of basis*
We also define a **change of basis**, from the basis $\phi : P_2 \hookrightarrow \mathcal{D}$ to the basis $\psi : P_2 \hookrightarrow \mathcal{D}$, to be an isomorphism $U : D \rightarrow D$ satisfying $U(\phi(s)) = \psi(s)$ for all $s \in P_2$.

In dynamical algebras, as in the motivating theory of linear algebra, given a pair of bases $\mathcal{B}_1$, $\mathcal{B}_2$ the change of basis that takes $\mathcal{B}_1$ to $\mathcal{B}_2$ is immediate.

Theorem 11. Let $\mathcal{D} = End(N)$ be a dynamical algebra in a unique decomposition category $(\mathbf{C}, \oplus)$ and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be distinct bases for $\mathcal{D}$.

We use part **(iii)** of Definition 12 and characterise $\mathcal{B}_1$ and $\mathcal{B}_2$ by two distinct strong embeddings of P_2 into $\mathcal{D}$. We denote the images of the generators of P_2 under these embeddings by $\{p, q, p^{-1}, q^{-1}\}$ and $\{r, s, r^{-1}, s^{-1}\}$ respectively.

We may then write down an isomorphism U giving the change of basis, using the matrix representation of Theorem 9, by

$$U \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = \left(\begin{array}{cc} rp^{-1} & rq^{-1} \\ sp^{-1} & sq^{-1} \end{array} \right) \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{cc} pr^{-1} & ps^{-1} \\ qr^{-1} & qs^{-1} \end{array} \right)$$

That is, the isomorphism U is conjugation by the matrix

$$R = \left(\begin{array}{cc} pr^{-1} & ps^{-1} \\ qr^{-1} & qs^{-1} \end{array} \right)$$

and its inverse, given by the 'generalised-inverse/transpose'

$$R^{-1} = \left(\begin{array}{cc} rp^{-1} & rq^{-1} \\ sp^{-1} & sq^{-1} \end{array} \right)$$

Proof. It is almost immediate that $U : D \rightarrow D$ satisfies the condition on the embeddings of P_2 given in Definition 12. It remains to show that U is indeed an isomorphism. However, as U is defined by conjugation, it suffices to show that R^{-1} is indeed the global inverse of R - i.e. $R^{-1}R = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = RR^{-1}$. By direct calculation,

$$R^{-1}R = \left(\begin{array}{cc} rp^{-1}pr^{-1} + rq^{-1}qr^{-1} & rp^{-1}ps^{-1} + rq^{-1}qs^{-1} \\ sp^{-1}pr^{-1} + sq^{-1}qr^{-1} & sp^{-1}ps^{-1} + sq^{-1}qs^{-1} \end{array} \right)$$

$$= \left(\begin{array}{cc} r(p^{-1}p + q^{-1}q)r^{-1} & r(p^{-1}p + q^{-1}q)s^{-1} \\ s(p^{-1}p + q^{-1}q)r^{-1} & s(p^{-1}p + q^{-1}q)s^{-1} \end{array} \right) = \left(\begin{array}{cc} r.1.r^{-1} & r.1.s^{-1} \\ s.1.r^{-1} & s.1.s^{-1} \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

The proof that RR^{-1} is also the identity follows similarly, and depends on the strong embedding condition $r^{-1}r + s^{-1}s = 1$. □

Definition 14. *The canonical change of basis*
We refer to the above isomorphism as the **canonical change of basis** from $\mathcal{B}_1$ to $\mathcal{B}_2$, and write $U : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ when the context is clear.

Corollary 12. Given an element m of a dynamical algebra $\mathcal{D}$, together with a matrix representation (with respect to a basis $\mathcal{B}_1$) as $m = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then the image of m under the canonical change of basis $U : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is given by

$$U \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \begin{array}{c} rp^{-1}Apr^{-1} + rp^{-1}Bqr^{-1} \\ +rq^{-1}Cpr^{-1} + rq^{-1}Dqr^{-1} \end{array} & \begin{array}{c} rp^{-1}Aps^{-1} + rp^{-1}Bqs^{-1} \\ +rq^{-1}Cps^{-1} + rq^{-1}Dqs^{-1} \end{array} \\ \\ \begin{array}{c} sp^{-1}Apr^{-1} + sp^{-1}Bqr^{-1} \\ +sq^{-1}Cpr^{-1} + sq^{-1}Dqr^{-1} \end{array} & \begin{array}{c} sp^{-1}Aps^{-1} + sp^{-1}Bqs^{-1} \\ +sq^{-1}Cps^{-1} + sq^{-1}Dqs^{-1} \end{array} \end{pmatrix}$$

where r, s are the generators of an embedding of P_2 in $\mathcal{D}$ that specify the basis $\mathcal{B}_2$

Proof. This is given by expanding out the matrix multiplications given in Theorem 11 above. ☐

Another corollory is that every automorphism (that is, endomorphism defined by conjugation) of a dynamical algebra is a change of basis.

Corollary 13. Let $\mathcal{D} = End(N)$ be a dynamical algebra in a unique decomposition category $\mathbf{C}$, with a specified basis $\phi : P_2 \rightarrow \mathcal{D}$, or equivalently $p, q, p^{-1}, q^{-1} \in \mathcal{D}$ (using part **(iii)** of Definition 12), and let the isomorphism $\Gamma : \mathcal{D} \rightarrow \mathcal{D}$ be defined by, for all $d \in \mathcal{D}$

$$d \mapsto U^{-1}dU \quad \text{for some isomorphism } U \in \mathcal{D}$$

Then this isomorphism arises as the canonical change of basis from ϕ to $U \circ \phi$.

Proof. This is immediate from the 2×2 matrix representation of arbitrary $d \in \mathcal{D}$ as

$$d \leftrightarrow \begin{pmatrix} pdp^{-1} & pdq^{-1} \\ qdp^{-1} & qdq^{-1} \end{pmatrix}$$

and the definition of the canonical change of basis from

$$\{p \ , \ q \ , \ p^{-1} \ , \ q^{-1}\}$$

to

$$\{U^{-1}pU \ , \ U^{-1}qU \ , \ U^{-1}p^{-1}U \ , \ U^{-1}q^{-1}U\}$$

Explicit calculations give that this change of basis satisfies $\Gamma(d) = U^{-1}dU$, and hence as U was an arbitrary isomorphism in $\mathcal{D}$, our result follows. ☐

6.1. *Changes of basis in the Geometry of Interaction*

An analysis of the first Geometry of Interaction paper (J.-Y. Girard 1988(i)) produces three distinct embeddings of the polycyclic monoid P_2 into the dynamical algebra of Definition 5 — in our terminology, three distinct bases for this dynamical algebra.

(i) The first basis arises from the internalisation of disjoint union by the Cantor 'interleaving' map, $p(n) = 2n$, $q(n) = 2n + 1$, and the other bases are defined in terms of this.
(ii) The second is based on an internalisation ψ of the Cartesian product — giving $(1 \times_\psi p)$ and $(1 \times_\psi q)$ as generators of the second embedding.
(iii) The third embedding is also based on an internalisation of the Cartesian product, using $(p \times_\psi 1)$ and $(q \times_\psi 1)$ as generators

The first embedding (and the resulting *internalised disjoint union*) is used for the representation of the multiplicative linear logic connectives, and the second is used in the representation of the !( ) exponential. The third embedding is used in the structural rules, and in the representation of the ?( ) exponential.

Using the above theory, we may write down automorphisms that transform any one embedding into any other, as a change of basis. This gives a previously unobserved symmetry in the algebra of the Geometry of Interaction; however the logical interpretation – if any – remains obscure.

7. Diagonalisations as changes of basis in dynamical algebras

An important application of basis changes in linear algebra is to construct diagonalisations of matrices. For a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ over a vector space V, a *diagonalisation* is a linear isomorphism D satisfying

$$D^{-1}MD = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}$$

for some elements A', B'. We demonstrate how this notion of diagonalisation has a direct analogue in dynamical algebras, and provide a sufficient condition (and related construction) for diagonalising an element by a change of basis.

Definition 15. *Diagonalisation in dynamical algebras*
Let $\mathcal{D} = End(N)$ be a dynamical algebra in a unique decomposition category $\mathbf{C}, \oplus$, and let $\mathcal{B}$ be a basis for $\mathcal{D}$. A **diagonalisation** of an element r written in matrix form as

$$r = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a pair of matrices L', L satisfying

$$L' \begin{pmatrix} A & B \\ C & D \end{pmatrix} L = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \quad , \quad L \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} L' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for some U, V.

As in the linear algebra case, diagonalising of arbitrary matrices is not straightforward. However, given the description of changes of basis in Theorem 11, we can characterise matrices that are diagonalised by a change of basis:

Theorem 14. Let $\mathcal{D} = End(N)$ be a dynamical algebra in a unique decomposition category $(\mathbf{C}, \oplus)$, and let $\mathcal{B}$ be a fixed basis for $\mathcal{D}$, given as an embedding of P_2 with generators denoted by p, q, p^{-1}, q^{-1} and giving rise to an internalisation of $\oplus$ that we denote $\oplus_{\mathcal{B}} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$.

Also let $\mathcal{B}'$ be an alternative basis for $\mathcal{D}$ given as an embedding of P_2 with generators denoted by r, s, r^{-1}, s^{-1}, and denote the canonical change of basis (as in Theorem 11) by $U : \mathcal{B} \to \mathcal{B}'$. Similarly, we use the notation $\oplus_{\mathcal{B}'}$ for the internalisation of the tensor $\oplus$ by the basis $\mathcal{B}'$.

Our claim is that this the matrices that may be diagonalised by this change of basis are exactly those of the form

$$\begin{pmatrix} p(X \oplus_{\mathcal{B}} Y)p^{-1} & p(X \oplus_{\mathcal{B}} Y)q^{-1} \\ q(X \oplus_{\mathcal{B}} Y)p^{-1} & q(X \oplus_{\mathcal{B}} Y)q^{-1} \end{pmatrix}$$

Proof. Consider an arbitrary diagonal matrix $\Delta = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. Then the image of Δ under the isomorphism U^{-1} is given by

$$U^{-1}(\Delta) = \begin{pmatrix} pr^{-1}Xrp^{-1} + qr^{-1}Yrq^{-1} & pr^{-1}Xrq^{-1} + ps^{-1}Ysq^{-1} \\ qr^{-1}Xrp^{-1} + qs^{-1}Ysp^{-1} & qr^{-1}Xrq^{-1} + qs^{-1}Ysq^{-1} \end{pmatrix}$$

Now observe that the internalisation of the tensor $\oplus$ by the basis $\mathcal{B}'$ may be written in terms of the embedding of P_2 generated by r, s, r^{-1}, s^{-1} as

$$f \oplus_{\mathcal{B}'} g \;=\; r^{-1}fr + s^{-1}gs$$

and so $U^{-1}(\Delta)$ may be written as

$$\begin{pmatrix} p(X \oplus_{\mathcal{B}} Y)p^{-1} & p(X \oplus_{\mathcal{B}} Y)q^{-1} \\ q(X \oplus_{\mathcal{B}} Y)p^{-1} & q(X \oplus_{\mathcal{B}} Y)q^{-1} \end{pmatrix}$$

Therefore all matrices of this form are diagonalised by the canonical change of basis $U : \mathcal{B} \to \mathcal{B}'$. Conversely, as Δ was chosen arbitrarily, all matrices diagonalised by the canonical change of basis $U : \mathcal{B} \to \mathcal{B}'$ are of this form. $\square$

8. Arbitrary finite matrices in Dynamical Algebras

Throughout the previous section, we have used the isomorphism between a dynamical algebra $\mathcal{D} = End(N)$, and the algebra of 2×2 matrices over $\mathcal{D}$ — that is, the isomorphism $N \cong N \oplus N$. Clearly, in the monogenic category $\mathbf{N}^{\oplus}$, we have isomorphisms between arbitrary objects, $X \cong Y$ for all $X, Y \in \mathbf{N}^{\oplus}$, and as $\mathbf{N}^{\oplus}$ is taken to be a UDC, by proposition 4, there exists isomorphisms between $\mathcal{D}$ and $(k \times k)$ matrices over $\mathcal{D}$, for arbitrary $k \geq 1$.

The 2×2 matrices were presented first, as this is a special case — it is unique in that we do not need to consider associativity isomorphisms:

Lemma 15. Let $\mathcal{D} = End(N)$ be a dynamical algebra in a Unique Decomposition Category $\mathcal{C}$. Then the associativity isomorphism

$$t : N \oplus (N \oplus N) \to (N \oplus N) \oplus N$$

can never be the identity map, and so $\mathcal{C}$ cannot be strictly associative.

Proof. We refer to (P. Hines 2003) for this proof, phrased in terms of self-similar objects in arbitrary monoidal categories. This argument in turn is derived from Isbell's proof that we need to consider associativity *up to isomorphism* in monoidal categories, as presented in (S. MacLane 1971). $\square$

We now demonstrate a systematic construction of the canonical associativity isomorphism between two distinct bracketings of n copies of a self-similar object N.

Proposition 16. Let $\mathcal{D} = End(N)$ be a dynamical algebra in a UDC, $(\mathbf{C}, \oplus)$, and let $\phi : N^{\oplus} \to \mathcal{D}$ be a self-embedding functor specifying a basis for $\mathcal{D}$. Then an object $X \in Ob(\mathbf{N}^{\oplus})$, given as a binary bracketing of k copies of N, specifies a strong embedding of the k^{th} polycyclic monoid $\gamma : P_k \to \mathcal{D}$, giving an isomorphism between $\mathcal{D}$ and the algebra of $k \times k$ matrices over $\mathcal{D}$.

Proof. We have a decomposition of X as the monoidal tensor of k objects (precisely, k copies of N); hence, as $\mathbf{C}$ is a UDC, there exist quasi-projections and quasi-inclusions, $\pi_i : X \to N$ and $\iota_i : N \to X$ satisfying

$$\pi_i \iota_j = \begin{cases} 1_N & i = j \\ 0_N & i \neq j \end{cases} \quad , \quad \sum_{i=1}^{k} \iota_i \pi_i \;=\; 1_X$$

The embedding $\gamma : P_k \to \mathcal{D}$ is generated by

$$\gamma(p_i) = \phi(\pi_i) \ , \quad \gamma(p_j^{-1}) = \phi(\iota_j) \ \ \forall \ 1 \leq i, j \leq k$$

and it is almost immediate that this is a strong embedding of P_k into $\mathcal{D}$.

This determines a $k \times k$ matrix representation of $\mathcal{D}$, since, by Proposition 4 an arrow $F : X \to X$ has a matrix representation as

$$F = \begin{pmatrix} F_{11} & \dots & F_{1k} \\ \dots & & \dots \\ F_{21} & \dots & F_{22} \end{pmatrix},$$

where $F_{ij} = \pi_i F \iota_j$. Hence the image of F under the self-embedding functor $\phi : \mathbf{N}^{\oplus} \to \mathbf{End(N)}$ is given by $\phi(F) = \sum_{i,j=1}^{k} p_j^{-1} F_{ij} p_i$ and the matrix representation follows from the isomorphism between X and N specified by the self-embedding functor ϕ. Therefore, any arrow $g \in \mathcal{D}$ has a matrix representation as

$$\begin{pmatrix} g_{11} & \cdots & g_{1k} \\ \dots & & \dots \\ g_{k1} & \cdots & g_{kk} \end{pmatrix} \quad \text{where } g_{ij} = p_i g p_j^{-1}$$

It may also be verified by direct calculation that this is indeed an isomorphism, as in the ring-theoretic calculations of (P. Hines, M.V. Lawson 1998). $\square$

We have shown that any binary bracketing of k copies of a self-similar object N corresponds to a strong embedding of P_k into $\mathcal{D} = End(N)$. We now demonstrate that given two distinct bracketings of k copies of N, the canonical associativity isomorphism between them arises naturally via a change of basis between the two distinct corresponding embeddings of P_k.

Theorem 17. Let $\mathcal{D} = End(N)$ be a dynamical algebra in a UDC, $(\mathbf{C}, \oplus)$, and fix $\phi : N^{\oplus} \to \mathcal{D}$, a self-embedding functor giving a basis for $\mathcal{D}$. Also, let $X, Y \in Ob(\mathbf{N}^{\oplus})$, be distinct binary bracketings of k copies of N.

By MacLane's coherence theorem (S. MacLane 1971), there exists a unique associativity, or rebracketing, isomorphism $T : X \to Y$, and by Proposition 16 above, these two objects specify two strong embeddings $\alpha, \beta : P_k \to \mathcal{D}$.

MacLane's associativity isomorphism may be written in terms of the strong embeddings of P_k, by

$$T = \begin{pmatrix} \alpha(p_1^{-1})\beta(p_1) & \alpha(p_1^{-1})\beta(p_2) & \dots & \alpha(p_1^{-1})\beta(p_k) \\ \alpha(p_2^{-1})\beta(p_1) & \alpha(p_2^{-1})\beta(p_2) & \dots & \alpha(p_2^{-1})\beta(p_k) \\ \dots & \dots & & \dots \\ \alpha(p_k^{-1})\beta(p_1) & \alpha(p_k^{-1})\beta(p_2) & \dots & \alpha(p_k^{-1})\beta(p_k) \end{pmatrix}$$

Proof. Recall that MacLane's proof of his coherence theorem was in terms of the *monogenic category* $\mathbf{W}$ specified by all binary bracketings of a single symbol w, together with a unique arrow between objects of the same rank (i.e. occurences of the symbol w).

In the monogenic category $\mathbf{N}^{\oplus}$ specified by a self-similar object in a UDC, once we have fixed a basis (that is, an isomorphism $N \cong N \oplus N$), we have an embedding of the monogenic category $\mathbf{W}$ — the unique arrow between objects x and y arises as the composite of the compression and division isomorphisms (as in Definition 8) $c : x \to N$ and $d : N \to Y$. Hence we may 'split' each canonical associativity isomorphism into a unique (up to the fixed choice of basis) composite of a compression and division map.

By the same reasoning as the proof of Lemma 10, the compression isomorphism for the object X is given in terms of the self-embedding functor ϕ, applied to the quasi-projections and quasi-inclusions. By Theorem 16 above, this gives

$$c_X = \begin{pmatrix} \alpha(p_1^{-1}) & \alpha(p_2^{-1}) & \dots & \alpha(p_k^{-1}) \end{pmatrix}$$

Similarly, the division isomorphism for an object Y is given by

$$d_Y = \begin{pmatrix} \beta(p_1) \\ \beta(p_2) \\ \dots \\ \beta(p_k) \end{pmatrix}$$

Expanding this product gives

$$T = d_Y c_X = \begin{pmatrix} \alpha(p_1^{-1})\beta(p_1) & \alpha(p_1^{-1})\beta(p_2) & \dots & \alpha(p_1^{-1})\beta(p_k) \\ \alpha(p_2^{-1})\beta(p_1) & \alpha(p_2^{-1})\beta(p_2) & \dots & \alpha(p_2^{-1})\beta(p_k) \\ \dots & \dots & & \dots \\ \alpha(p_k^{-1})\beta(p_1) & \alpha(p_k^{-1})\beta(p_2) & \dots & \alpha(p_k^{-1})\beta(p_k) \end{pmatrix}$$

as required. □

Discussion It is immediate from the matrix decompositions allowed by UDCs, and the arbitrary isomorphisms between self-similar objects in the monogenic category $N^{\oplus}$, that we have arbitrary $(k \times k)$ matrix representations of a dynamical algebra $\mathcal{D} = End(N)$. However, we might naively assume that we need to describe the bases for a $(k \times k)$ matrix decomposition explicitly, for each $k \in \mathbb{N}$.

What the above shows, in combination with Corollary 13, is that it is enough to present a single self-embedding functor $\phi : N^{\oplus} \to End(N)$ (equivalently, isomorphism $N \cong N \oplus N$, or strong embedding of P_2 into $\mathcal{D} = End(N)$). Although a basis, in the sense of Definition 12, is not enough to characterise $k \times k$ matrix representations of a dynamical algebra $\mathcal{D}$, it is enough to characterise them *up to associativity*, in the precise sense of MacLane's coherence theorem.

9. The countable infinite case

Given the close connection between strong embeddings of the polycyclic monoid P_k into a dynamical algebra $\mathcal{D}$, and $k \times k$ matrix algebras, we now study embeddings of the countably infinite polycyclic monoid P_∞.

Lemma 18. Let $\mathcal{D}$ be a dynamical algebra. Then there exists an embedding of P_∞ into $\mathcal{D}$.

Proof. By Theorem 8 there exists an embedding of P_2 into $\mathcal{D}$. However, a standard result on polycyclic monoids (M. Nivat, J.-F. Perrot 1970) states that there exists embeddings of all countable polycyclic monoids into P_2. Hence our result follows. □

Discussion Recall that all strong embeddings $P_k \hookrightarrow \mathcal{D}$, for finite K, arise as the images of quasi-projections / quasi-inclusions under a self-embedding functor on a UDC. However, the definition of UDCs (Definition 6) only allows for *finite* quasi-projections / quasi-inclusions. Hence, in order to consider analogues of countably infinite matrices over a dynamical algebra, we need to axiomatise the required properties.

Definition 16. *Infinite matrices, matrix algebras*
Let $\mathcal{D} = End(N)$ be a dynamical algebra in a UDC $(\mathbf{C}, \oplus)$. We define an infinite matrix over $\mathcal{D}$ to be a countable subset of $\mathcal{D}$, doubly indexed by two copies of the natural numbers (- i.e. terms of the form $m = \{m_{ij}\}_{i,j \in \mathbb{N}}$). We denote the set of infinite matrices over $\mathcal{D}$ by $Mat_\infty(\mathcal{D})$, and define the **composition** of matrices to be a *partial* function given by $[m \circ n]_{ki} = (gf)_{ki} = \sum_{j=1}^{\infty} m_{kj} n_{ji}$, and this composite of two matrices is defined exactly when the above sum exists, for all $k, i \in \mathbb{N}$.

We define a **matrix algebra** to be a subset A of $Mat_\infty(\mathcal{D})$ satisfying:

— $m, n \in A$ implies $m \circ n$ is well-defined, and is in A.
— There exists some matrix $I \in A$ satisfying $I \circ a = a = a \circ I$, for all $a \in A$.

In a matrix algebra, by definition, the partial composition is a global function. By convention we denote it by juxtaposition, and reserve the $\circ$ notation for when it is potentially partial.

Infinite matrix algebras over a dynamical algebra $\mathcal{D} = End(N)$ are *not* required to be endomorphism monoids (or submonoids) of some 'infinitary object' $\Omega \in Ob(\mathbf{C})$. In particular unique decomposition categories are not required to have arbitrary countable monoidal tensors.

We now demonstrate the connection between infinite matrix algebras and strong embeddings of the polycylic monoid P_∞ — however, in a UDC, the condition $\sum_{i=0}^{k} \iota_i \pi_i = 1_N$ for quasi-projections/inclusions at a self-similar object only guarantees the existence of strong embeddings of all *finite* polycylic monoids. Although Lemma 18 gives embeddings of P_∞, there is no guarantee that these are strong (i.e. $\sum_{i=0}^{\infty}$ exists and is the identity).

Theorem 19. Let $\mathcal{D} = End(N)$ be a dynamical algebra in a UDC $(\mathbf{C}, \oplus)$, and let $\gamma : P_\infty \to \mathcal{D}$ be a strong embedding. Then $\mathcal{D}$ has a representation as an infinite matrix algebra over $\mathcal{D}$.

Proof. (We abuse notation, and denote $\gamma(w) \in \mathcal{D}$ by $w \in \mathcal{D}$ for all $w \in P_\infty$).

The representation Φ_γ is defined in terms of the embedding of P_∞ by, for arbitrary $f \in \mathcal{D}$, $[\Phi_\gamma(f)]_{ij} = p_i f p_j^{-1}$. To see that $im(\Phi_\gamma)$ is an infinite matrix algebra, and Φ_γ is also a homomorphism, observe that

$$([\Phi_\gamma(g)] \circ [\Phi_\gamma(f)])_{ki} = \sum_{j=0}^{\infty} [\Phi_\gamma(g)]_{kj} [\Phi_\gamma(f)]_{ji}$$

and by definition,

$$[\Phi_\gamma(g)]_{kj} = p_k g p_j^{-1} \quad , \quad [\Phi_\gamma(f)]_{ji} = p_j f p_i^{-1}$$

Therefore,

$$([\Phi_\gamma(g)] \circ [\Phi_\gamma(f)])_{ki} = \sum_{j=0}^{\infty} p_k g p_j^{-1} p_j f p_i^{-1} = p_k g \left(\sum_{j=0}^{\infty} p_j^{-1} p_j \right) f p_i^{-1}$$

However, as the embedding $\gamma : P_\infty \to \mathcal{D}$ is strong, $\sum_{j=0}^{\infty} p_j^{-1} p_j = 1$, and so

$$([\Phi_\gamma(g)] \circ [\Phi_\gamma(f)])_{ki} = p_k (gf) p_i^{-1} = [\Phi_\gamma(gf)]_{ki}$$

Also, the identity of $\mathcal{D}$ under the Φ_γ homomorphism acts as an identity for this matrix algebra.

If we denote the image of Φ_γ by $\mathcal{A}$, we may define an inverse by $\Phi_\gamma^{-1} : \mathcal{A} \to \mathcal{D}$ by $\Phi_\gamma^{-1}(m) = \sum p_i^{-1} m_{ij} p_j$, for all $m \in \mathcal{A}$. For an element $f \in \mathcal{D}$,

$$\Phi_\gamma \Phi_\gamma^{-1}(f) = \sum_{i,j=0}^{\infty} p_j^{-1} p_j f p_i^{-1} p_i = 1.f.1 = f$$

Finally, since composition in a dynamical algebra distributes over summation, we may define a component-wise summation on $\mathcal{A}$, and $\Phi_\gamma \left(\sum_{i \in I} f_i \right) = \sum_{i \in I} \Phi_\gamma(f_i)$. □

Definition 17. *Infinite matrix representations of dynamical algebras*
Given a dynamical algebra $\mathcal{D} = End(N)$, and a strong embedding of an infinite polycylic monoid $\gamma : P_\infty \to \mathcal{D}$, we refer to the image of $\Phi_\gamma : \mathcal{D} \to Mat_\infty(\mathcal{D})$ as the **infinite**

matrix representation of $\mathcal{D}$ determined by γ. We use the notation $Mat_\gamma(\mathcal{D})$ for this matrix algebra.

Proposition 20. Let $\mathcal{D} = End(N)$ denote a dynamical algebra, and let $\gamma, \theta : P_\infty \to \mathcal{D}$ be distinct strong embeddings. Then there exists a unique isomorphism from $C_{\gamma,\theta} : Mat_\gamma(\mathcal{D}) \to Mat_\gamma(\mathcal{D})$.

Proof. From Theorem 19 above, we have isomorphisms $\Phi_\gamma^{-1} : Mat_\gamma(\mathcal{D}) \to \mathcal{D}$ and $\Phi_\theta : \mathcal{D} \to Mat_\theta(\mathcal{D})$. The isomorphism $C_{\gamma,\theta} : Mat_\gamma(\mathcal{D}) \to Mat_\gamma(\mathcal{D})$ arises as the composite:

$$\begin{tikzcd}
 & \mathcal{D} \arrow[dr, "\Phi_\theta"] & \\
Mat_\gamma(\mathcal{D}) \arrow[ur, "\Phi_\gamma^{-1}"] \arrow[rr, "C_{\gamma,\theta}"'] & & Mat_\theta(\mathcal{D})
\end{tikzcd}$$

□

Observation The isomorphism $C_{\gamma,\theta}$ above plays a similar rôle to the rebracketing $T : X \to Y$ isomorphism of Theorem 17, and the change of basis isomorphism of Theorem 11. However, we are not always able to write $C_{\gamma,\theta}$ explicitly as an analogous infinite matrix, although it is well-defined as an isomorphism of infinite matrix algebras.

Proposition 21. Given a dynamical algebra $\mathcal{D} = End(N)$, a strong embedding of an infinite polycylic monoid $\gamma : P_\infty \to \mathcal{D}$, and a summable ordered list $(l_0, l_1, l_2, \ldots)$ of elements of $\mathcal{D}$, then the infinite matrix

$$L = \begin{pmatrix} l_0 & 0 & 0 & \ldots \\ 0 & l_1 & 0 & \\ 0 & 0 & l_2 & \\ \ldots & & & \ldots \end{pmatrix}$$

is a member of $Mat_\gamma(\mathcal{D})$.

Proof. This is almost immediate from the summability of the list $(l_0, l_1, l_2, \ldots)$, and the identity $\Phi_\gamma(L) = \sum_{i=0}^{\infty} \gamma(p_i^{-1}) l_i \gamma(p_i)$. □

A number of dynamical algebras (including those of Definition 5) satisfy a much stronger property, that has a close connection with the ! exponential of linear logic:

Definition 18. *Exponential dynamical algebras, Girard's bang, diagonalisability*
Given a dynamical algebra $\mathcal{D} = End(N)$, we say that it is an **exponential dynamical algebra** when there exists a strong embedding of an infinite polycylic monoid $\gamma : P_\infty \to \mathcal{D}$ such that the matrix

$$L = \begin{pmatrix} l_0 & 0 & 0 & \ldots \\ 0 & l_1 & 0 & \\ 0 & 0 & l_2 & \\ \ldots & & & \ldots \end{pmatrix}$$

is a member of $Mat_\gamma(\mathcal{D})$, for arbitrary $\{l_i\} \subseteq \mathcal{D}$.

The terminology ‘exponential’ is motivated by the representation of Girard’s Bang !( ) operation in the Geometry of Interaction, as the infinite diagonal matrix

$$!(f) = \left(\begin{array}{cccc} f & 0 & 0 & \dots \\ 0 & f & 0 & \\ 0 & 0 & f & \\ \dots & & & \dots \end{array} \right)$$

Exponential dynamical algebras, and the existence of such ‘infinite diagonal matrices’, allows us to consider infinitary analogues of the diagonalisation procedure of Section 6.1: we say that a matrix $m \in Mat_\gamma(\mathcal{D})$ is **diagonalisable** if there exists an injective homomorphism $\Delta : Mat_\gamma(\mathcal{D}) \rightarrow Mat_\theta(\mathcal{D})$ such that $\Delta(m)$ is a diagonal matrix.

10. List representations and manipulations using matrix algebras

Given infinite matrices over an algebra, there are very natural ways to represent lists and list manipulations. We show that these may also take place within dynamical algebras:

Definition 19. *Dyadic lists, Convolved lists*
Let $\mathcal{D} = End(N)$ be a dynamical algebra.

We define the **dyadic list monoid** over $\mathcal{D}$ to be the set of all lists

$$Dyad(\mathcal{D}) = \{(l_0, l_1, l_2, \dots) : l_i \in \mathcal{D}\}$$

together with the natural pointwise composition

$$(k_0, k_1, k_2, \dots)(l_0, l_1, l_2, \dots) = (k_0 l_0, k_1 l_1, k_2 l_2, \dots)$$

It is immediate that this is closed under composition, and is monoid, since the infinite list $(1, 1, 1, \dots)$ acts as an identity.

We also define the **convolved list monoid** to be to be the set of all summable lists

$$Conv(\mathcal{D}) = \left\{ (l_0, l_1, l_2, \dots) : \sum_{i=0}^{\infty} l_i \in \mathcal{D} \right\}$$

together with the convolution product $(kl)_c = \sum_{c=b+a} k_b l_a$.

It is immediate from the requirement that all lists are summable that this is well-defined, and the composition is associative. Note that this is always a monoid, with identity given by the list $(1, 0, 0, 0, \dots)$.

Proposition 22. Let $\mathcal{D} = End(N)$ be an exponential dynamical algebra, and let $\gamma : P_\infty \rightarrow \mathcal{D}$ be a strong embedding. Then there exists injective monoid homomorphisms

$$\delta : Dyad(\mathcal{D}) \rightarrow Mat_\gamma(\mathcal{D}) \quad , \quad \kappa : Conv(\mathcal{D}) \rightarrow Mat_\gamma(\mathcal{D})$$

Proof.
The homomorphism δ is given by

$$[\delta(l_0, l_1, l_2, \dots)]_{ij} = \left\{ \begin{array}{ll} l_i & i = j \\ 0 & i \neq j \end{array} \right.$$

It is immediate that this is injective, and by Proposition 21, it is in the image of $\Phi_\gamma : \mathcal{D} \to Mat_\gamma(\mathcal{D})$. It is also trivial that this is a homomorphism, by definition of matrix multiplication.

The homomorphism κ is given by

$$[\kappa(l_0, l_1, l_2, \ldots)] = \begin{cases} l_{j-i} & i \geq j \\ 0 & i < j \end{cases}$$

It is immediate that this is also an injection. The requirement that the list $(l_0, l_1, \ldots)$ is summable also implies that it is in the image of Φ_γ. To see that it is a monoid homomorphism, observe that the definition of matrix multiplication gives

$$[\kappa(k_0, k_1, \ldots)\kappa(l_0, l_1, \ldots)]_{ij} = \begin{cases} h_{j-i} & i \geq j \\ 0 & i < j \end{cases} \quad \text{where } h_c = \sum_{a+b=c} k_b l_a \quad \forall\ c \in \mathbb{N}$$

as required. Finally,

$$[\kappa(1, 0, 0, \ldots)]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

which is the identity for the infinite matrix algebra $Mat_\gamma(\mathcal{D})$. □

Questions The image of $\delta : Dyad(\mathcal{D}) \to Mat_\gamma(\mathcal{D})$ is exactly the diagonal elements of $Mat_\gamma(\mathcal{D})$. Given the theory of diagonalisation and changes of basis:

1 When can elements of $im(\kappa)$ be diagonalised by (a) an injection, or (b) a change of basis?

2 What about when we use a two-sided version of κ (for 2-sided lists with a convolution product), such as

$$[K(\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots)]_{ij} = \begin{cases} l_{j-i} & i \geq j \\ l_{i-j} & j \geq i \end{cases}$$

11. Acknowledgements

There are a great many people I wish to thank for their input and interest in this paper: Samson Abramsky, for helpful comments and suggestions of where definitions and results needed to be formalised, Bob Coecke and Ross Duncan for interest and patience working through preliminary versions, Mark Lawson for results on polycylic monoids and the algebraic theory of self-similarity, Prakash Panangaden for motivating the idea of bases by the observation that self-embedding functors are naturally only up to isomorphism, and Philip Scott for helpful discussions on the structure of UDCs and enrichments of categories over notions of summation in general (including the Σ-groups of Appendix A).

Appendix A: When are linear algebras dynamical?

A natural question is how the structures of dynamical algebras relate to the usual notions of matrix decomposition, diagonalisation, bases, &c. for linear algebras. Surprisingly, for structures so closely related to linear algebra, the categories of Hilbert or Banach spaces are *not* UDCs. This is because of the *positivity property* for Σ-monoids, given in (E. Manes, M. Arbib 1986):

Proposition 23. Positivity of Σ-monoids
Let S be a Σ-monoid, and $\{x_i\}_{i\in I}$ be a summable family. Then

$$\sum_{i\in I} x_i = 0 \quad \Leftrightarrow \quad x_i = 0 \ \forall i \in I$$

Proof. Let X=$\{x_i\}_{i\in I}$ be a summable family satisfying $\sum_{i\in I} x_i = 0$. For some $i \in I$, we define $Y = \{x_j\}_{j\neq i\in I}$, so $x_i + \sum Y = 0$. By the partition-associativity axiom,

$$x_i = x_i+(\sum Y+x_i)+(\sum Y+x_i)+(\sum Y+x_i)+\ldots = (x_i+\sum Y)+(x_i+\sum Y)+\ldots = 0$$

Hence $x_i = 0$. However, as i was chosen arbitrarily, $x_k = 0$ for all $k \in I$. □

The positivity property is an entirely natural property for computational systems such as the Geometry of Interaction, where an overall description of a computation is built up from the ‘addition’ of subsystems (as in the standard trace formula of (E. Haghverdi 2000)). However, in order to consider linear algebras in these terms, we either need to restrict ourselves to *positive cones* over linear algebras, as in (P. Selinger 2004), or use a different notion of summability.

An appropriate alternative notion of summability is to consider categories that are enriched over Σ-*Groups*, rather than Σ-monoids, equipped with quasi-projections and quasi-inclusions. Σ-Groups were introduced, in slightly different terms, in (S. Wylie 1957); however, we follow the notations and conventions of (D. Higgs 1988), in particular because of the direct application to Banach spaces given in (D. Higgs 1989).

Definition 20. *Series, subseries, and contractions of Abelian groups, sober series, Σ-Groups*
Let A denote an abelian group. A **series** over A is a countably indexed family $x = \{x_i\}_{i\in I}$. Series over A have a **addition** $+$ given by disjoint union, and **inverses**, $-$, given by $-x = \{-x_i\}_{i\in I}$. A **subseries** x' of a series $x = \{x_i\}_{i\in I}$ is specified by some subset $I' \subseteq I$, giving $x' = \{x_i\}_{i\in I'}$.

Given a series $\{x_i\}_{i\in I}$ over an Abelian group A, and a partition of x into finite subsets, $I = \{I_j\}_{j\in J}$, then as A is abelian, and I_j is finite, for all $j \in J$, we may define

$$y_j = \sum_{i\in I_j} x_i$$

We say that the series $y = \{y_j\}_{j\in J}$ is a **contraction** of x.

A series x is called **sober** when there does not exist a contraction z of x in which some $a \neq 0 \in A$ occurs infinitely often.

A **Σ-Group** is defined to be an abelian group A, together with a set S of series over A, and an infinite summation operation $\Sigma : S \to A$, satisfying, for all $a \in A$, and $x, y \in S$,

— All series $s \in S$ are sober.
— $\{a\} \in S$, and $\sum\{a\} = a$.
— $x + (-y) \in S$, and $\sum(x - y) = (\sum x) - (\sum y)$.

— Given $s \in S$, and s' a contraction of s, then $s' \in S$, and $\sum s = \sum s'$.

Proposition 24. Let **Banach** denote the category of continuous linear maps between Banach spaces. Then for any objects R, S, the hom-set $\mathbf{Banach}(R, S)$ is a Σ-Group.

Proof. We refer to (D. Higgs 1989) for this proof, along with a detailed investigation of the structure of these Σ-groups. ☐

In order to consider analogues of dynamical algebras for categories of linear maps, we axiomatise UDC-like structures that are enriched over Σ-groups, rather than Σ-monoids.

Definition 21. *Group-like UDCs, Group-like dynamical algebras*
We define a **group-like UDC** (or **gUDC**) to be a symmetric monoidal category $(\mathbf{C}, \oplus)$ enriched over Σ-groups, satisfying the following condition:

— For any object $\bigoplus_{i=1}^n X_i$, there exist **quasi-projections** and **quasi-inclusions**

$$\pi_k : \bigoplus_{i=1}^{n} X_i \to X_k \ , \ \iota_k : X_k \to \bigoplus_{i=1}^{n} X_i$$

satisfying $\pi_i \iota_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$ and $\sum_{i=1}^n \iota_i \pi_i = 1_{\oplus_{i=1}^n X_i}$

As a direct analogy of the UDC case, we define a **group-like dynamical algebra**, or **GD-algebra** to be the endomorphism monoid of a self-similar object in a gUDC.

Lemma 25. Countably infinite dimensional spaces in the category $(\mathbf{Hilb}, \oplus)$ of Hilbert spaces with direct sum, are self-similar objects, and hence their endomorphism monoids are GD-algebras.

Proof. Recall that Hilbert spaces L, S over some field $\mathcal{K}$, that satisfy $dim(L) = dim(S)$ are isomorphic. Therefore, a countably infinite dimensional space N is isomorphic to $N \oplus N$, and hence is a self-similar object in this category. To demonstrate that it gives a GD-algebra, note that the direct sum is a biproduct, and so we have projections and inclusions satisfying the required condition. ☐

Proposition 26. Let $U : H \to H$ be an isomorphism from a countably infinite dimensional Hilbert space to itself. Then U is uniquely specified by two distinct embeddings of the Cuntz algebra into $End(H)$.

Proof. It has been observed many times (or may be verified by direct calculation) that the Cuntz algebra (J. Cuntz 1977) is a representation of P_2 as linear maps on an infinite-dimensional space; it is, in our terminology, a strong embedding of P_2. Therefore, the monoid of linear maps from H to itself, enriched over the Σ-group structure of the hom-sets, is a GD-algebra. Now observe that Corollory 13 is equally applicable to GD-algebras, and so every automorphism of this monoid is uniquely determined by a canonical change of basis. However, it is a classical result on Hilbert spaces that every endomorphism arises as an automorphism, and hence our result follows. ☐

Although we can perform many of the same algebraic constructions for both dynamical algebras and GD-algebras, they are entirely disjoint algebraic and categorical structures; Σ-groups are not special cases of Σ-monoids, nor vice versa. The question of whether there is a structure with partial addition, weak enough to contain both Σ-groups and Σ-monoids, but strong enough to allow for matrix algebra and Geometry of Interaction style constructions remains open.

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