A short note on coherence and self-similarity

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This paper is dedicated to Max Kelly, whose work on categorical coherence has been a great inspiration, and source of understanding.

Abstract

MacLane's original introduction to the theory of monoidal categories presented a short argument, due to J. Isbell, of why the concept of 'associativity up to isomorphism' is needed for a reasonable conception of a monoidal tensor. This argument was based on the properties of a distinguished object D in a category with a product, satisfying $D = D \times D$. In the following paper, we demonstrate that a slight modification of this property allows us to construct elements of $End(D)$ that have similar properties to associativity isomorphisms in a monoidal category, and show how these can be used to construct what can reasonably be considered to be a weakening of the associativity of a strict monoidal category.

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1 Introduction

In his introduction to the theory of monoidal categories, [5] p.160, MacLane gives an argument, due to Isbell, of why a non-strict version of associativity (that is, associative up to isomorphism) is needed. This justification was phrased in terms of a denumerable set D in the category of functions on sets. He demonstrated how the identity $D = D \times D$, together with a strictly associative product, implies $f = g = h$ for arbitrary $f, g, h : D \to D$. Continuing from this observation, it was demonstrated that allowing a tensor to be associative *up to isomorphism* would allow for a reasonable (i.e. non-trivial) theory.

In the following, we invert this argument, and demonstrate that strict associativity, and a non-strict version of *self-similarity* (that is, the identity $D = D \times D$) allows us to take a category with a strict monoidal tensor, and replace it with an 'equivalent' category that has a non-strict monoidal tensor.

This process can be thought of as a routine method of 'weakening' the strict associativity of a monoidal category.

2 Self-similarity and one-object monoidal structures

The case of a (non-strict) monoidal category with a specified object N satisfying $N \cong N \otimes N$ has been studied extensively in [3], so we do not consider it here. Instead, we consider a weaker version of what [3] refers to as 'self-similarity'.

Definitions 1 *Let* $\mathbf{M}, \otimes$ *be a monoidal category. We say that an object* $N \in Ob(\mathbf{C})$ *is* weakly self-similar *if* $N \otimes N$ *is a retract of* N *— that is, if there exist arrows* $c : N \otimes N \to N$*, and* $d : N \to N \otimes N$ *satisfying* $dc = 1_{N \otimes N}$ *and* $cd = e : N \to N$ *where* $e^2 = e \neq 1$*. For the purposes of this paper, we are interested in the special case where the monoidal tensor* $\otimes$ *is strict.*

The canonical example of this is the category of partial maps on Sets, together with the disjoint union; we define N to be the unit interval, d to be the map that takes two copies of $[0,1]$ to the left and right thirds of the unit interval, and c to be its (partially defined) inverse. Clearly, $dc = 1_{[0,1] \sqcup [0,1]}$, $cd = 1_{[0,\frac{1}{3}] \cup [\frac{2}{3},1]}$, and iterating the c and d maps will give successive approximations in the intuitive construction of the Cantor set. Other examples can be constructed from the natural numbers in the category of partial bijections, using any of the familiar bijections between $\mathbb{N}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{N} \sqcup \mathbb{N}$, and then composing this with an injection from $\mathbb{N}$ to a strict (denumerable) subset $X \subset \mathbb{N}$.

The endomorphism monoids of weakly self-similar objects then have a structure that is similar to (but weaker than) a monoidal structure, as follows:

Lemma 1 *Let* N *be a weakly self-similar object of a (strict) monoidal category* $\mathbf{M}, \otimes$*, and denote the endomorphism monoid of* N *by* S*, for clarity. Then there exists a semigroup homomorphism* $\odot : S \times S \to S$*, and an element* $t \in S$ *satisfying*

- $t((f \odot g) \odot h) = (f \odot (g \odot h))t$
- $(t \odot 1)t(1 \odot t) = t^2$
- $t \neq 1$

Proof 1 *We define* $f \odot g = c(f \otimes g)d$*, for all* $f, g \in S$*. Then* $\odot$ *is a semigroup homomorphism, since*

$$(f \odot g)(h \odot k) = c(f \otimes g)dc(h \otimes k)d = c(f \otimes g)(h \otimes k)d$$

$$= c(fh \otimes gk)d = (fh \odot gk)$$

However, $\odot$ is not a monoid homomorphism, since $1 \odot 1 < 1$, by construction.

We then define $t = c(1_N \otimes c)(d \otimes 1_N)d : N \to N$. The intuitive reasoning behind this definition can be seen if we (temporarily) denote the associativity isomorphism of $\mathbf{M}$ (which is the identity map) by α, and draw t as the following composite:

$$\begin{array}{ccccc} N & \xrightarrow{d} & N \otimes N & \xrightarrow{d\otimes 1} & (N \otimes N) \otimes N \\ \downarrow{\scriptstyle t} & & & & \downarrow{\scriptstyle \alpha = 1_{N\otimes N \otimes N}} \\ N & \xleftarrow{c} & N \otimes N & \xleftarrow[1\otimes c]{} & N \otimes (N \otimes N) \end{array}$$

From the definition of $\odot$, it is immediate that $t((f \odot g) \odot h) = (f \odot (g \odot h))t$, and either direct calculation, or the observation that this is the unique arrow satisfying $t(1\odot(1\odot 1)) = ((1\odot 1)\odot 1)t$ will demonstrate that $(t\odot 1)t(1\odot t) = t^2$.

Finally, to see that t cannot be the identity, recall Isbell's argument, from [5]; this can easily be extended to the case when the function from $D \times D$ to D is a retract, rather than an identity. $\Box$

3 Constructing a non-strict version of a strict monoidal functor

Definitions 2 *Let $\mathbf{C}, \otimes$ be a strict monoidal category, and let N be a weakly self-similar object, as defined in Definitions 1. We study the full subcategory of $\mathbf{C}$ generated by the object N and the tensor $\otimes$. This category, which we refer to as $\mathbf{N}^{\otimes}$ has all the structure of a symmetric monoidal category apart from the unit object I.*

This category is a strict monoidal category (apart from the unit). We demonstrate that the structure of $S = End(N)$ allows us to construct a non-strict version of $\mathbf{N}^{\otimes}$:

Theorem 2 *The Karoubi envelope of the monoid S contains a subcategory generated by an object isomorphic to N, and a non-strict tensor.*

Proof 2 *The Karoubi envelope of a monoid S, denoted $\mathbf{K_S}$, is defined to be the category whose objects are the idempotents of the monoid S, and whose arrows are given by $a \in \mathbf{K_S}(e, f)$ iff $fae = a \in S$. This is used in [1] as a way of constructing typed logical systems from untyped systems.*

Consider the full subcategory of $\mathbf{K_S}$ whose objects are given by all the idempotents generated by the identity and the semigroup homomorphism $\odot$. We denote this subcategory by $\mathbf{1}^{\odot}$. Then it is almost immediate that $\mathbf{1}^{\odot}$ has the *structure of a monoidal category (apart from the units) — this can be seen by taking the monoidal tensor of* $\mathbf{1}^{\odot}$ *to be the extension of* $\odot$ *to* $\mathbf{K_S}$*, and defining the associativity element from* $(e \odot (f \odot g))$ *to* $((e \odot f) \odot g$ *to be the element of* S *given by* $((e \odot f) \odot g)t(e \odot (f \odot g))$*.*

Finally, note that this monoidal structure is not strict, so the associativity elements are never identities, and it is trivial from the construction that the endomorphism monoid of 1 *in this category is isomorphic to the endomorphism monoid of* N *in the category* $\mathbf{M}$. □

The justification for considering this category to be a non-strict version of the category $\mathbf{N}^{\otimes}$ comes from noting that if the monoidal tensor of M is non-strict, then this construction will give an isomorphism between $\mathbf{N}^{\otimes}$ and $\mathbf{1}^{\odot}$. Therefore, this construction can be thought of as a method of 'weakening the strict associativity of a monoidal functor'.

4 Historical context

The above construction is clearly based on the Geometry of Interaction representation of linear logic, [2] (where an embedding of the 'dynamical algebra' into the monoid of relations on the natural numbers gives an injective function from $\mathbb{N} \sqcup \mathbb{N}$ to $\mathbb{N}$, and is then used to construct analogues of symmetric monoidal structures in a monoid, as shown in [3]).

Lambek and Scott also give an application of the Karoubi envelope in terms of weak C-monoids, using the identity of the monoid to give a self-similar object in the Karoubi envelope in [1] p. 100.

However, the original motivation for this construction comes from an analysis of the technique of 'Halving Projections' in the K-theory of certain C^* algebras, [4], as follows:

The set of all matrices over a unitary ring R form a strict monoidal category, with the usual coproduct $A \sqcup B = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$. If we consider the calculation of the K_0 group of $B(l^2)$, the 'halving projections' technique demonstrates that every idempotent matrix over l^2 is related to an idempotent in $B(l^2)$ (using the usual equivalence relation in K_0 theory, being $e \sim f$ iff there exists x, y satisfying $xy = e$, $yx = f$, $exf = x$, $fye = e$). This relation is a ring-theoretic version of the maps giving a retract from N to $N \otimes N$, and so we can extend it to all matrices over $B(l^2)$, and use it to define an analogue $\odot$ of the coproduct of matrices — however, note that the coproduct of matrices is strict, and the one-object analogue $\odot$ is strict up to isomorphism (as given by the retraction process).

The next step in the calculation of the K_0 group of $B(l^2)$ is to construct $\sim$ equivalence classes of idempotents of $B(l^2)$, and by the definition of this equivalence, this clearly identifies $(e \odot (f \odot g))$ with $((e \odot f) \odot g)$.

From here, it is immediate that this quotient forces strict associativity, and a strict isomorphism from the matrices over $B(l^2)$ to $B(l^2)$. However, we have already seen that this will force all elements to be the identity, by Isbell's argument. Hence, there is a basic categorical interpretation of the triviality of the K_0 group of $B(l^2)$.

References

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