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The Structure of Partial Isometries

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Abstract

It is well known that the ‘quantum logic’ approach to the foundations of quantum mechanics is based on the subspace ordering of projectors on a Hilbert space. In this paper, we show that this is a special case of an ordering on partial isometries, introduced by Halmos and McLaughlin. Partial isometries have a natural physical interpretation, however, they are notoriously not closed under composition. In order to take a categorical approach, we demonstrate that the Halmos-McLaughlin partial ordering, together with tools from both categorical logic and inverse categories, allows us to form a category of partial isometries.

This category can reasonably be considered a ‘categorification’ of quantum logic — we therefore compare this category with Abramsky & Coecke’s ‘compact closed categories’ approach to foundations, and with the ‘monoidal closed categories’ view of categorical logic. This allows us to decide both whether these approaches to foundations are compatible, and whether ‘quantum logic’ can be interpreted in categorical logic terms.

1.1 Introduction

As early as 1936, von Neumann and Birkhoff proposed treating projectors on Hilbert space as propositions about quantum systems [BvN36], by direct analogy classical order-theoretic approaches to logic. Boolean lattices arise as the Lindenbaum-Tarski algebras of propositional logics, and as the set of all projectors on a Hilbert space also forms an orthocomplemented lattice, the operations *meet*, *join*, and *complement* were

analogously interpreted as the logical connectives *conjunction*, *disjunction* and *negation*.

However, the lattice of projectors is not a Boolean lattice, so this interpretation requires modifications to the rules of propositional logic (notably the distributive law, $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ fails, and is replaced by the weaker condition $A \leq C \Rightarrow A \wedge (A^\perp \vee C) = C$). The resulting system of *orthomodular lattices* has become known as *quantum logic*, and a number of authors [DF62, HP69] have suggested that the non-classical behaviour of quantum systems simply results from the fact that orthomodular lattices, rather than Boolean lattices, provide the ‘natural logic’ of such systems.

This paper does not take a position on this claim, although we do discuss arguments for and against a logical interpretation of orthomodular lattices. Instead, we consider a natural order-theoretic generalisation, where the dynamics of quantum systems may also be viewed in order-theoretic terms.

The lattice of projectors on a Hilbert space is an inherently static view of a quantum system. In the usual treatment, the dynamics of a system is interpreted in terms of operations on this lattice — for example, a unitary map induces an automorphism of the orthomodular lattice of projectors. However, a specific subspace may be considered as providing partial information about a quantum system, whereas an automorphism is an inherently global operation. Thus the ‘static’ view provided by quantum logic is based on partial information, whereas the ‘dynamics’ is based on a global view.

In this paper we study combinations of dynamical processes (considered as unitary maps) and measurements (considered as projectors), from both an order-theoretic and category-theoretic viewpoint. To this end, we study *partial isometries*. These generalise both projectors and unitaries in a natural way, and we demonstrate in Corollary 1.3.7 that partial isometries on a finite-dimensional space may be characterised as the composite of a unitary map and a projector (the infinite-dimensional case is, not unexpectedly, more complex). Partial isometries also have a natural partial order introduced by Halmos and McLaughlin in [HMcL63], and when restricted to projectors this is exactly the partial order of the orthomodular lattice of ‘quantum propositions’.

Using this partial order, we study partial isometries from a category-theoretic point of view. It is well-known [IE68] that the composite of two partial isometries is not, in general, a partial isometry. However, there is a natural associative composition (based on the conjunction of

quantum logic, and closely related to the treatment of partial isometries in inverse semigroup theory) that allows us to define a category of partial isometries. The resulting category is shown to be an *inverse category*. Inverse categories also have a natural partial order on their hom-sets — this is exactly the Halmos-McLaughlin partial order and hence, when restricted to projectors, the orthomodular lattice ordering of quantum logic.

Given this ‘categorification’ of quantum logic, a natural question is then: *what are the similarities and differences between the resulting categorical structures, and Abramsky & Coecke’s ‘categorical foundations’ program for quantum mechanics [AC05] ?*

We demonstrate that superficially there is a good case to be made for agreement between the two categorical approaches, but despite this, detailed calculations shows that they are *incompatible*. However, the reason for this incompatibility is somewhat unsatisfactory; the categorical foundations approach is based on compact closure, as an abstract version of post-selected teleportation. However, a treatment of this cannot be given in the category of partial isometries, due to its inability to express post-selection. We further demonstrate that no possible agreement can be found; as an application of general categorical principles (i.e., that arbitrary limits should be preserved by the adjunction demonstrating categorical closure) we show that the category of partial isometries is not closed, never mind compact closed. From a categorical logic point of view, this is the statement that the categorification of quantum logic does not have a well-behaved implication.

1.2 The order theory of projectors, and ‘quantum logic’

The order theory of projectors on Hilbert space is the foundation of the ‘quantum logic’ of Birkhoff / von Neumann [BvN36]. The partial order on projectors is defined as follows:

Definition 1.2.1 The lattice of projectors on a Hilbert space:

*Let $E, F : H \rightarrow H$ be projectors on a Hilbert space H . We say that E is **below** F , written $E \leq F$ when $EF = E$. Note that this implies $EF = FE$. It is straightforward that \leq is a partial order, and the set of all projectors on H forms a lattice, with top element the identity map $\top = 1_H$ and bottom element the zero map $\perp = 0_H$.*

The meet and join of this lattice may be given explicitly:

Proposition 1.2.2 Partial orders and meets on projectors:

Let E, F be projectors on some Hilbert space, corresponding to the subspaces H_E, H_F respectively. Then

- (i) The join $E \vee F$ is defined by

$$E \vee F = \text{Inf}\{G : E \leq G \text{ and } F \leq G\}$$

and is simply the projector onto the smallest subspace containing both H_E and H_F .

- (ii) The meet $E \wedge F$ is defined by

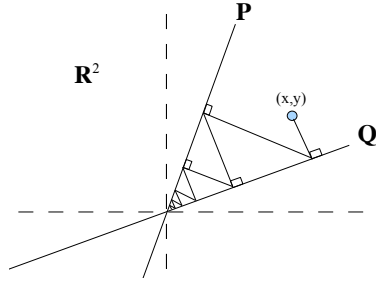
$$E \wedge F = \text{Sup}\{G : G \leq E \text{ and } G \leq F\}$$

and is the projector onto the largest subspace contained within both H_E and H_F . It may be given explicitly by $E \wedge F = \lim_{N \rightarrow \infty} (EF)^N = \lim_{n \rightarrow \infty} (FE)^N$.

Proof We refer to [JJ86] for these results, and [AS70] for a proposed physical interpretation of the meet operation. \square

We illustrate the above characterisation of the meet, with two 1-dimensional subspaces P, Q , in Figure 1.1.

Fig. 1.1. The meet of two projectors, in \mathbb{R}^2



Lattices of projectors on Hilbert space are shown in [CP76] to be *orthomodular lattices*, defined as follows:

Definition 1.2.3 orthomodular lattices, orthocomplemented lattices:

Let L, \leq be a complete lattice. We say that it is **orthocomplemented** when there exists an involution $()^\perp : L \rightarrow L$ that satisfies

- (i) $A \vee A^\perp = \top$.
(ii) $A \wedge A^\perp = \perp$.

(iii) $A \leq B$ if and only if $B^\perp \leq A^\perp$.

An orthocomplemented lattice L is called **orthomodular** if it satisfies the additional condition

(4) $A \leq B$ implies $B = A \vee (A^\perp \wedge B)$

The canonical example of an orthomodular lattice is the lattice of projectors on a Hilbert space (in fact, we refer to [AG57] for a stronger statement). The orthocomplement of the projector onto a subspace $S \leq H$ is simply the projector onto the orthogonal complement $S^\perp = \{h \in H : \langle s|h \rangle = 0, \forall s \in S\}$.

Orthomodular lattices as logics Following a long-established tradition in order theory, ‘quantum logic’ treats the elements of a lattice of projectors as propositions, and *meet* and *join* as *conjunction* and *disjunction* respectively (although note the failure of the distributive law described in Section 1.1). The involution is interpreted as negation, and satisfies analogues of de Morgan’s laws $(P \wedge Q)^\perp = P^\perp \vee Q^\perp$.

At this point, readers familiar with mathematical logic will naturally wonder about *implication*. By analogy with order-theoretic approaches to classical logic, there are 5 distinct candidates for an implication. The most commonly studied is the *Sasaki hook*, which may be defined in terms of conjunction, disjunction and negation as $P \overset{S}{\rightarrow} Q = P^\perp \vee (P \wedge Q)$. As we will discuss in Section 1.7.1, it is controversial whether this should be accepted as a genuine implication. We refer to [GH81] for properties of the Sasaki hook, and a strong defence of this connective as a form of implication.

Quantum logic and foundations of quantum mechanics We emphasise that we are not taking a position on ‘quantum logic’ as a foundation for quantum mechanics. Rather, we consider a natural order-theoretic generalisation that introduces a notion of partial dynamics as an intrinsic part of this order theory, in order to study quantum logic from a categorical viewpoint. For an introduction to quantum mechanics via quantum logic, we refer to [RH89] for an excellent exposition. We also refer to [RG02] for the related, but conflicting, ‘consistent histories’ approach to foundations, and to [SV89, SA91] for connections between order theory, logic, and computation in the classical world.

1.3 Partial Isometries

As stated in the introduction, the objective of this paper is to introduce a notion of partial dynamics to the lattice of projectors on Hilbert spaces. We do this by considering both the order theory and category theory of partial isometries. We first present the definitions, and various simple properties.

The following definitions are taken from [HMCL63]

Definition 1.3.1 partial isometries, initial and final subspaces, isometries:

Let $L : H_1 \rightarrow H_2$ be a linear map of Hilbert spaces, and denote its adjoint by $L^* : H_2 \rightarrow H_1$. Then L is a **partial isometry** when $L^*L : H_1 \rightarrow H_1$ is a projector, and hence (or equivalently) $LL^* : H_2 \rightarrow H_2$ is a projector.

The projectors $E_L = L^*L : H_1 \rightarrow H_1$ and $F_L = LL^* : H_2 \rightarrow H_2$ are called the **initial** and **final projectors** of L , and the corresponding subspaces are the **initial** and **final subspaces**. When the initial subspace is the whole of H_1 , then L is called an **isometry**. There is no standard terminology for the adjoint of an isometry, where the final subspace is the whole of H_2 .

Given a partial isometry L , it is immediate that L^* is also a partial isometry, and the initial projector of L is the final projector of L^* . Both unitary maps and projectors are trivially partial isometries. The initial and final projectors of a unitary map are the global identities on its source and target space, and a projector is its own initial and final projector.

1.3.1 Basic properties

We establish some basic algebraic results on partial isometries:

Proposition 1.3.2 Standard results on partial isometries

Given partial isometries $L : H_1 \rightarrow H_2$, $M : H_2 \rightarrow H_3$, $N : K_1 \rightarrow K_2$, then:

- (i) L is a unitary map between its initial and final subspaces.
- (ii) $LL^*L = L$ and $L^*LL^* = L^*$
- (iii) When the initial and projectors of $L : H_1 \rightarrow H_2$ are the global identities of H_1 and H_2 respectively, then L is a unitary map.

- (iv) $ML : H_1 \rightarrow H_3$ is a partial isometry exactly when the initial projector of M commutes with the final projector of L , so $E_M F_L = F_L E_M$.
- (v) $L \oplus K : H_1 \oplus K_1 \rightarrow H_2 \oplus K_2$ is a partial isometry.
- (vi) $L \otimes K : H_1 \otimes K_1 \rightarrow H_2 \otimes K_2$ is a partial isometry.

Proof Results (1)-(4) are taken from [IE68]), and Results (5) and (6) are a simple consequence of linearity. Note that (2) states that the adjoint $(\)^*$ is a *generalised inverse* (in the sense of semigroup theory [MVL88]) on partial isometries. However, the set of all partial isometries on a Hilbert space H is neither a regular nor an inverse semigroup, since by (4), it is not closed under composition. \square

Corollary 1.3.3 *The composite of a unitary and a partial isometry is always a partial isometry.*

Proof The initial and final projectors of a unitary map $U : H \rightarrow K$ are the global identities on H, K respectively. The result then follows trivially from (4) of Proposition 1.3.2 above. \square

1.3.2 The Halmos-McLaughlin partial order

We now show that the partial order on projectors given in Section 1.2 is a special case of the partial order on partial isometries given by Halmos & McLaughlin in [HMcL63] :

Definition 1.3.4 The Halmos-McLaughlin partial order

The partial order \leq on partial isometries is defined in [HMcL63] by

$$L \leq K \Leftrightarrow L = KE_L$$

or equivalently,

$$L \leq K \Leftrightarrow L = F_L K$$

i.e. K , when restricted to the initial subspace of L , or co-restricted to the final subspace of L , is exactly L . It is then immediate that \leq , when restricted to projectors, is exactly the orthomodular lattice partial ordering of Definition 1.2.1. We refer to \leq as the **Halmos-McLaughlin partial order**, or **HML partial order**.

The projectors on a space H may be characterised as ‘*partial isometries beneath the identity 1_H* ’. From the physical interpretation as composites of unitaries and projectors, we also have a particular interest in partial isometries that are beneath unitary maps:

Definition 1.3.5 Physical partial isometries

Given a partial isometry $L : H \rightarrow H$ satisfying $L \leq U$ for some unitary map $U : H \rightarrow H$, we refer to L as a **physical partial isometry**.

Proposition 1.3.6 Let $L : H \rightarrow H$ be a partial isometry. When the codimension of the initial subspace is equal to the codimension of the final subspace, then L is a physical partial isometry.

Proof Denote the initial subspace of $L : H \rightarrow H$ by S , and the terminal subspace by T , so $S \oplus S^\perp = H = T \oplus T^\perp$. By definition of partial isometries $\dim(S) = \dim(T)$, and the condition on the codimensions gives that $\dim(S^\perp) = \dim(T^\perp)$, and so $S^\perp \cong T^\perp$. Given a (not necessarily unique) unitary $L' : S^\perp \rightarrow T^\perp$ exhibiting this isomorphism, we may construct a unitary $U = L + \begin{pmatrix} 0 & 0 \\ 0 & L' \end{pmatrix} : S \oplus S^\perp \rightarrow T \oplus T^\perp$, and it is immediate from the definition that $L \leq U$. \square

Corollary 1.3.7 All partial isometries on finite-dimensional spaces are physical isometries.

Proof The condition on codimensions from Proposition 1.3.6 is trivially satisfied for partial isometries between finite-dimensional spaces. Counterexamples on infinite-dimensional spaces include the Cuntz-Krieger algebras of [CK80], and the Shift operator of [JG76]. \square

Interpretation Note that the definition of a *physical partial isometry* is restricted to the case where the source and target space are the same. In this case, a physical partial isometry is simply one that may be ‘completed’ to a unitary map.

In general, we may often give a physical interpretation to partial isometries where the source and target space differ; as a simple example, for a norm-1 vector $\phi \in H$, the bra and ket operators $\langle \phi | : H \rightarrow \mathbb{C}$ and $|\phi\rangle : \mathbb{C} \rightarrow H$ are both partial isometries. These simple examples will become important when considering the structure of the category of partial isometries, in Section 1.8.

1.4 The interaction of partial isometries and projectors

We now introduce a useful technique for dealing with the interaction of partial isometries and projectors. This is strongly based on a technique from inverse semigroup theory (see, for example, [JH95]), although the partial isometries on a space H do not form a semigroup.

Proposition 1.4.1 “Pushing a projector through a partial isometry.”

Let $L : H_1 \rightarrow H_2$ be a partial isometry, and let $G : H_1 \rightarrow H_1$ and $D : H_2 \rightarrow H_2$ be projectors satisfying $G \leq E_L$ and $D \leq F_L$. Then

- (i) there exists a unique projector $G' \leq F_L$ such that $LG = G'L$.
- (ii) there exists a unique projector $D' \leq E_L$ such that $DL = LD'$.

Proof

- (i) Define $G' : H_2 \rightarrow H_2$ by $G' = LGL^*$. Then $(G')^* = (LGL^*)^* = LGL^*$, so G' is self-adjoint. Similarly,

$$G'G' = LGL^*LGL^* = LGE_LGL^* = LGGL^* = LGL^* = G'$$

so G' is idempotent, and hence it is a projector. Now note

$$G'L = LGL^*L = LGE = LEG = LL^*LG = LG$$

as required. To show that $G' \leq F_L$, note that

$$\begin{aligned} G'F &= G'LL^* = LGL^*LL^* = LGEL^* \\ &= LEL^* = LL^*LL^* = F^2 = F \end{aligned}$$

Uniqueness follows since L is a unitary when restricted to its initial / final subspaces.

- (ii) Defining $D' : H_1 \rightarrow H_1$ by $D' = L^*DL$, this result follows by symmetry. □

This ‘pushing a projector through a partial isometry’ operation is order-preserving, as shown:

Lemma 1.4.2 Let $L : H_1 \rightarrow H_2$ be a partial isometry, and let $P, Q : H_2 \rightarrow H_2$ be projectors below F_L . Then the unique projectors $P', Q' \leq E_L$ satisfying

$$PL = LP' \quad , \quad QL = LQ'$$

satisfy $P \leq Q \Leftrightarrow P' \leq Q'$.

Proof (\Rightarrow) Assume $P \leq Q$, so $PQ = QP = P$. By construction $P' = L^*PL$ and $Q' = L^*QL$ so

$$P'Q' = L^*PLL^*QL = L^*LL^*PQL = L^*PQL = L^*PL = P'$$

and hence $P' \leq Q'$.

(\Leftarrow) This proof is almost identical to (\Rightarrow) above. \square

The technique of ‘pushing a projector through a partial isometry’ allows us to establish a connection between partial isometries and block matrices, as follows:

Proposition 1.4.3 *Let $U : H \rightarrow K$ be a unitary map, let $H = H_1 \oplus \dots \oplus H_a$ and $K = K_1 \oplus \dots \oplus K_b$ be direct sum decompositions of the source and target space. We may then write U as a $(b \times a)$ block matrix*

$$U = \begin{pmatrix} U_{11} & \dots & U_{1a} \\ \vdots & \ddots & \vdots \\ U_{b1} & \dots & U_{ba} \end{pmatrix}$$

where $U_{ij} : H_j \rightarrow K_i$ for all $1 \leq i \leq b$ and $1 \leq j \leq a$.

For fixed i, j , the submatrix

$$B_{ij} = \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} \\ \vdots & U_{ij} & \vdots \\ \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}$$

is given by $B_{ij} = QUP$ where $Q = \mathbf{0} \oplus 1_{K_i} \oplus \mathbf{0} : K \rightarrow K$ and $P = \mathbf{0} \oplus 1_{H_j} \oplus \mathbf{0} : H \rightarrow H$. This is then a partial isometry exactly when

$$Q'P = PQ' \quad \text{or equivalently,} \quad QP' = P'Q$$

where $Q' = U^{-1}QU$ and $P' = UPU^{-1}$ are given by ‘passing the projector P (resp. Q) through the unitary U ’, as in Proposition 1.4.1.

Proof Given $B_{ij} = QUP : H \rightarrow K$, then since U is unitary, $F_U = 1_K$, and $Q \leq F_U$. Therefore, by Proposition 1.4.1, $QU = UQ'$, and this is a partial isometry with initial projector Q' . Hence, by Proposition 1.3.2, $QUP = UQ'P$ is a partial isometry exactly when $Q'P = PQ'$, as required. The equivalent condition $QP' = P'Q$ follows either algebraically (by conjugation by U), or by duality. \square

Interpretation We may consider a quantum computation that consists of a finite series of operations — either unitary maps, or measurements† on certain subspaces. A natural question would be whether this is equivalent to a series of unitaries (and by composition, a single unitary map) followed by a measurement. However, a unitary followed by a projector is a partial isometry, whereas a series of unitaries and projectors is not, in general. Proposition 1.4.3 gives conditions for a projector, followed by a unitary, followed by a projector to be a partial isometry (i.e. equivalent to a unitary followed by a projector), and this is easily generalised to a series of unitaries and measurements.

1.5 A category of partial isometries

From Proposition 1.3.2, the class of partial isometries is *not* closed under the usual composition of linear maps. However, with a modified composition (that we demonstrate is equivalent to a construction of [MVL88]), partial isometries are not only closed under composition, but form an *inverse category*.

Definition 1.5.1 Inverse Categories, Generalised Inverses

*Inverse categories are the natural extension of inverse monoids to the many-object case [MVL88]. A category \mathbf{C} is **inverse** when for every arrow $f \in \mathbf{C}(X, Y)$, there exists a unique **generalised inverse** $f^{-1} \in \mathbf{C}(Y, X)$ satisfying*

$$ff^{-1}f = f \quad \text{and} \quad f^{-1}ff^{-1} = f^{-1}$$

*We emphasise that this axiom does not imply $f^{-1}f = 1_X$ or $ff^{-1} = 1_Y$. Generalised inverses that satisfy these additional conditions are called **left** and **right global inverses** respectively.*

Interpretation The usual computer science interpretation of inverse categories is strictly stronger than simply requiring reversibility. Rather, inverse categories are used (as in [PH97, EH00, AHS02, HS08]) to model resource-sensitive systems, where copying and deleting are either forbidden, or strictly controlled (such as the *Linear Logic* of [JYG87a]). When modelling quantum information, we would also expect similar structures, due to the no-cloning and no-deleting theorems [WZ82, PB00].

† For simplicity, we assume that these measurements are *post-selected* — if the desired measurement outcome is not observed, we abandon the experiment and start again.

In order to define a category of partial isometries, we give a binary operation on partial isometries that we will prove is the composition in a category:

Definition 1.5.2 *Given partial isometries $L : H_1 \rightarrow H_2$ and $M : H_2 \rightarrow H_3$, we define*

$$M \circ L = \lim_{n \rightarrow \infty} [(ML)(ML)^*]^n (ML) = \lim_{n \rightarrow \infty} (ML) [(ML)^*(ML)]^n$$

For readers familiar with Girard's Geometry of Interaction system [JYG87b, JYG90], this definition is clearly motivated by the *execution formula*†. However, it has an even closer connection with the conjunction in quantum logic:

Lemma 1.5.3 *Given partial isometries $M : H_2 \rightarrow H_3$, $L : H_1 \rightarrow H_2$, as above, then $M \circ L = M(E_M \wedge F_L)L : H_1 \rightarrow H_3$ where E_M and F_L are the final and initial projectors of M and L respectively. Hence $M \circ L$ exists for arbitrary partial isometries $L : H_1 \rightarrow H_2$ and $M : H_2 \rightarrow H_3$, and is a partial isometry.*

Also, when $E_M F_L = F_L E_M$, then $M \circ L$ is simply ML , the usual composition of M and L as linear maps.

Proof From Proposition 1.2.2,

$$E_M \wedge F_L = \lim_{n \rightarrow \infty} (E_M F_L)^n = \lim_{n \rightarrow \infty} (F_L E_M)^n$$

For arbitrary $n \geq 1$, $[(ML)(ML)^*]^n (ML) = (MLL^*M^*)^n ML$, and rebracketing gives

$$[(ML)(ML)^*]^n (ML) = M(LL^*M^*M)^n L = M(F_L E_M)^n L$$

Hence

$$\lim_{n \rightarrow \infty} [(ML)(ML)^*]^n (ML) = \lim_{n \rightarrow \infty} M(F_L E_M)^n L = M(E_M \wedge F_L)L$$

As $E_M \wedge F_L$ is a projector that commutes with both E_M and F_L , it follows from Proposition 1.3.2 that:

† As a historical note, the original Geometry of Interaction system was presented in terms of partial isometries acting on infinite-dimensional Hilbert spaces. However — as noted by many authors [SA96, AHS02, EH00, PH97] — the partial isometries used were of a very special form. Precisely, all the action takes place within the image of Barr's injective functor [MB82] from Partial Injections into Hilbert spaces, $l_2 : \mathbf{pInj}^{op} \rightarrow \mathbf{Hilb}$. Thus, although partial isometries were used, all initial and all final projectors commute. This is clearly a very restricted case, and in that sense the framework of Hilbert spaces and partial isometries can be considered as 'useless superstructure'.

- (i) $(E_M \wedge F_L)L : H_1 \rightarrow H_2$ is a partial isometry,
- (ii) $M(E_M \wedge F_L) : H_1 \rightarrow H_2$ is a partial isometry,
- (iii) and hence $M(E_M \wedge F_L)L : H_1 \rightarrow H_3$ is a partial isometry.

Finally, when $E_M F_L = F_L E_M$, then $E_M \wedge F_L = E_M F_L = F_L E_M$, and so $M \circ L = ML$. \square

Theorem 1.5.4 *Partial isometries, with the composition given above, form an inverse category.*

Proof It is shown in [MVL88] that the set of all partial isometries acting on a single space, together with this composition is an inverse monoid. The extension to the many-object case is immediate. \square

Notation We denote the category of partial isometries with the above composition by **pIsom**. By contrast, we denote the category of continuous linear maps on Hilbert spaces (with the usual composition) by **Hilb**. We will use the notation \circ for the composition in **pIsom**, and simply use concatenation to denote the composition in **Hilb**. In both cases, we have a particular interest in the case where we restrict to finite-dimensional spaces. We denote these restrictions by **Hilb_{FD}** and **pIsom_{FD}**.

Not only is the category **pIsom** closed under the composition \circ , but there is a very strong sense in which the composition $M \circ L$ can be thought of as a supremum within the HML partial ordering, as follows:

Proposition 1.5.5 *Let $M : H_2 \rightarrow H_3$ and $L : H_1 \rightarrow H_2$ be partial isometries, and let $P : H_2 \rightarrow H_2$ be a projector such that*

$$MP : H_2 \rightarrow H_3 \quad \text{and} \quad PL : H_1 \rightarrow H_2 \quad \text{and} \quad MPL : H_1 \rightarrow H_3$$

are all partial isometries. Then $MPL \leq M \circ L$, where \leq is the Halmos-McLaughlin partial order of Definition 1.3.4.

Proof Since MP and PL are partial isometries, from Proposition 1.3.2, $PE_M = E_M P$ and $PF_L = F_L P$. Hence $Q = E_M P F_L$ is a projector satisfying $MPL = MQL$. We now work with this projector Q . By construction, $Q \leq E_M$ and $Q \leq F_L$, so by definition $Q \leq (E_M \wedge F_L)$. Now consider the unique projectors $Q', R : H_1 \rightarrow H_1$ satisfying $MQL = MLQ'$ and $M(E_M \wedge F_L)L = MLR$ given as in Proposition 1.4.1. From

Lemma 1.4.2, we deduce $Q' \leq R$, so $Q'R = Q'$, and so $MLRQ' = MLQ'$. However, by definition of the Halmos-McLaughlin partial order,

$$MPL = MQL = MLQ' \leq MLR = M(E_M \wedge F_L)L$$

as required. □

It is then straightforward to write down the initial and final projectors of $M \circ L$. These are closely related to the notion of ‘pushing a projector through a partial isometry’, as given in Proposition 1.4.1.

Corollary 1.5.6 *Let $L : H_1 \rightarrow H_2$ and $M : H_2 \rightarrow H_3$ be as above. Then*

- (i) *The initial projector of $M \circ L$ is the unique projector $P \leq E_L$ satisfying $(E_M \wedge F_L)L = LP$, as in Proposition 1.4.1.*
- (ii) *The final projector of $M \circ L$ is the unique projector $Q \leq F_M$ satisfying $QM = M(E_M \wedge F_L)$, as in Proposition 1.4.1.*

Proof The initial projector P of $M(E_M \wedge F_L)L$ may be given explicitly by:

$$P = L^*(E_M \wedge F_L)M^*M(E_M \wedge F_L)L = L^*(E_M \wedge F_L)E_M(E_M \wedge F_L)L$$

However, $(E_M \wedge F_L) \leq E_M$ and $(E_M \wedge F_L)^2 = E_M \wedge F_L$, so $P = L^*(E_M \wedge F_L)L$. The final projector Q may similarly be shown to be $Q = M(E_M \wedge F_L)M^*$.

Results (1) and (2) then follow by comparing these explicit forms with Proposition 1.4.1. □

1.6 The inverse structure of \mathbf{pIsom}

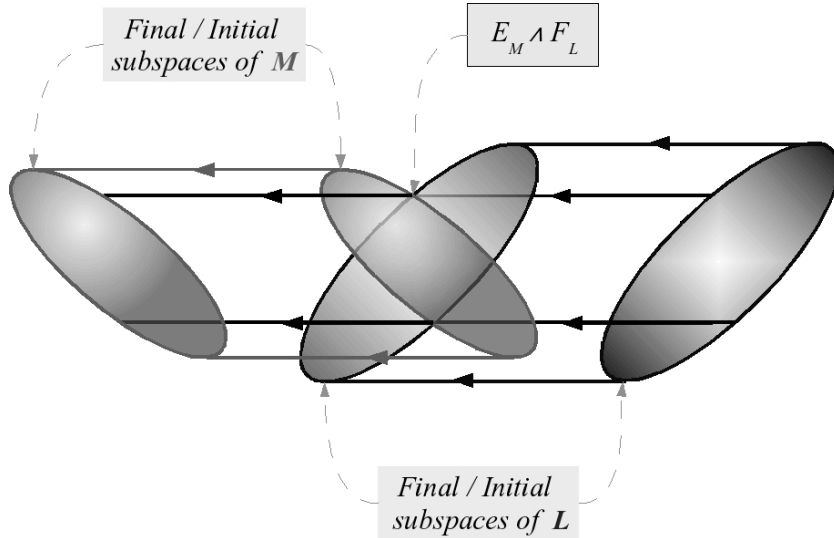
From the characterisation of the composition given in Proposition 1.5.3, and the inverse category theoretic result of 1.5.4, there is a simple graphical representation of the composition \circ , as shown in Figure 1.2.

We now consider the consequences of the inverse structure on partial isometries. An immediate corollary is the commutativity of projectors, as follows:

Corollary 1.6.1

- (i) *The idempotents of \mathbf{pIsom} are exactly the projectors of \mathbf{Hilb} .*

Fig. 1.2. The composition $M \circ L$ of partial isometries



- (ii) All projectors (at the same object) commute, so $E \circ F = F \circ E$.
- (iii) Given projectors $E, F : H \rightarrow H$, then $E \circ F = E \wedge F$.

Proof

- (i) Consider an idempotent $L \circ L = L$ in **pIsom**. Then $L = L(E_L \wedge F_L)L$, and hence $E_L \leq F_L$ and $F_L \leq E_L$. Therefore, $E_L = F_L$, and so $L \circ L = LL$, and L is thus idempotent in the category **Hilb**. Hence, as L is a partial isometry it is Hermitian, and so L is a projector.
- (ii) The commutativity of idempotents follows from the uniqueness of generalised inverses, as standard result of inverse semigroup theory [MP55].
- (iii) This is immediate from the definition of the composition \circ .

□

Commutativity of Projectors The commutativity of projectors in **pIsom** follows from the existence and uniqueness of generalised inverses, and therefore is an essential feature of this category. We note the strong

distinction with the ‘matrix mechanics’ formulation of quantum mechanics and the behaviour of projectors in the category **Hilb**, where the *non-commutativity of projectors* captures the ‘non-classical’ behaviour of observations [MR82].

1.6.1 Partial orders and inverse structures

So far, we have seen that the Halmos-McLaughlin partial order generalises the ‘quantum logic’ ordering of projectors on Hilbert space (Definition 1.3.4), and have used the conjunction operation of quantum logic to define a composition on partial isometries (Definition 1.5.2 and Lemma 1.5.3). This composition is then exactly the calculation of a supremum within the HML partial order (Proposition 1.5.5), and allows us to define an inverse category of partial isometries (Theorem 1.5.4).

However, every inverse semigroup has a partial order defined on its elements, as in [MVL88], and this has an immediate generalisation to the hom-sets of inverse categories:

Definition 1.6.2 The natural partial order of an inverse category
Let S be an inverse semigroup. The **natural partial order** \trianglelefteq on S is defined by

$$s \trianglelefteq t \Leftrightarrow \exists e^2 = e \ s = te$$

We refer to [MVL88] for proof that this is indeed a partial order, and its properties.

The generalisation of this notion to inverse categories is immediate. Let \mathcal{C} be an inverse category, and consider $f, g \in \mathcal{C}(X, Y)$. We extend the above definition in the obvious way, so

$$f \trianglelefteq g \Leftrightarrow \exists e^2 = e \in \mathcal{C}(X, X) \ f = ge$$

and this makes the hom-set $\mathcal{C}(X, Y)$ into a partial order.

Theorem 1.6.3 The Halmos-McLaughlin partial order \leq is exactly the natural partial order \trianglelefteq on the inverse category **pIsom**.

Proof Consider $L, M \in \mathbf{pIsom}(H, K)$.

- (\Rightarrow) Assume $L \leq K$. Then by definition, $L = ME_L$. Hence, as ME_L is a partial isometry, $ME_L = M(E_M \wedge E_L) = M \circ E_L$, so $L \trianglelefteq M$.

- (\Leftarrow) Assume $L \trianglelefteq M$, so there exists some $G^2 = G \in \mathbf{pIsom}(H, H)$ such that $L = M \circ G$. Therefore, $L \circ G = M \circ G \circ G = M \circ G = L$, and the initial subspace of L is a subspace of the initial subspace of M . Hence $L \leq M$, as required.

□

The natural partial order, and the generalised inverse Although there is a close connection between the generalised inverse and the natural partial order of an inverse category, we emphasise that the generalised inverse is *not* an orthocomplement. In particular, given $f \trianglelefteq g \in \mathcal{C}(X, Y)$, then $f^{-1} \trianglelefteq g^{-1} \in \mathcal{C}(Y, X)$. Not only is the generalised inverse not an orthocomplement, but $\mathbf{pIsom}(H, H)$ cannot in general be orthocomplemented since in general $L \not\leq 1_H$, so we do not even have a lattice structure[†].

We refer to [MVL88] for a comprehensive list of properties of \trianglelefteq from a semigroup-theoretic point of view.

1.7 Category theoretic structures, and ‘quantum logic’

Now we have established the structure of the category of partial isometries, we make an explicit comparison with the categorical structures used in the ‘categorical foundations’ program for quantum mechanics. We base this comparison on the logical interpretations in both cases.

We first summarise — without taking a position ourselves — various criticisms of the Birkhoff-von Neumann quantum logic program. This demonstrates that the controversial features (the treatment of the tensor product, and interpretation of implication) are exactly the elements taken as primitive in categorical logic generally.

We then give a very brief summary of the categorical approach to logic, via *closed categories*, and describe the connections between a particular form of categorical closure *compact closure*, and the quantum properties of entanglement and teleportation, that is the basis of the categorical foundations program [AC05].

Finally, we consider whether the category of partial isometries is compact closed, or indeed closed at all. This is in order to establish either a close connection, or irreconcilable difference, between the logical foun-

[†] This raises the natural question of what sort of order-theoretic structures are involved in the study of partial isometries. We briefly discuss this in Section 1.10. However, the emphasis of this paper is on the categorical properties.

dations in terms of orthomodular lattices, and the logical foundations in terms of compact closed categories.

1.7.1 Criticisms of Birkhoff / von Neumann's logical interpretations

As a general principle, the meet and the join of a lattice have a natural intuitive interpretation as conjunction and disjunction. This is the basis of interpretation of projectors as propositions in Birkhoff-von Neumann quantum logic. However, the treatment of implication is more controversial.

Implication in an orthomodular lattice is often defined in terms of the Sasaki hook, $P \overset{S}{\rightarrow} Q = P^\perp \vee (P \wedge Q)$, as an analogue of how implication may be defined in classical order-theoretic terms. The intuitive interpretation as a statement about ‘quantum propositions’ is not straightforward — we refer to [GH81, SS01, GH79] for various proposals. More seriously, several authors [HH83, GH81] have questioned whether it should be considered as an implication at all. This criticism is often based on the failure of the ‘deduction theorem’ of classical logic: $(a \wedge b) \leq c$ iff $a \leq (b \rightarrow c)$. This property is equivalent to distributivity in a lattice, and distributivity is exactly the point at which quantum logic diverges from classical logic — no connective on the lattice of projectors can satisfy this property [JM90]. Moreover, as described in [GH81], failure of the deduction theorem has serious implications, among which is the failure of *transitivity* (that is, $(A \Rightarrow B)$ and $(B \Rightarrow C)$ together imply $A \Rightarrow C$).

Another criticism of the orthomodular lattices approach is that the tensor product has no natural, purely order-theoretic, definition. Although we may certainly form the tensor product of two Hilbert spaces H, K , and interpret the lattice projections in the resulting space $H \otimes K$ as propositions about some compound system, there is no natural way of taking the lattice of projectors of H , and the lattice of projectors of K , and producing the lattice of projectors of $H \otimes K$. In order to use the tensor product (and hence reason about compound systems), we need to step outside lattice theory, use Hilbert space operations, and interpret the resulting lattice of projectors. This is a drawback in the program of providing foundations for quantum mechanics purely in lattice-theoretic, or logical, terms.

We now consider an alternative approach to logic generally, where

implication is primitive, and the conjunction (interpreted as a categorical tensor) may be defined in terms of its relationship to the implication.

“*Even the crows on the roofs caw about the nature of conditionals.*”
— Callimachus, Librarian of Alexandria, 300 BC, [KK71].

1.7.2 Logic, category-theoretically

In the order-theoretic approach to logic, implication is defined in terms of meet and join, interpreted as conjunction and disjunction. In stark contrast (although, see Section 1.7.2), many approaches to categorical logic take the notion of implication or deduction as primitive. For example, in a categorical setting for intuitionistic logic (such as [LS86]), an arrow $f : A \rightarrow B$ is treated, as per the Brouwer-Heyting-Kolmogorov interpretation [SU98], as a proof of proposition B from the assumption of proposition A .

The closure property of logic (that a proof of B from the assumption of A is equivalent to a proof of $A \Rightarrow B$) is taken as primitive, and modelled by *categorical closure* — a hom-set of arrows $\mathcal{C}(A, B)$ is itself an object of \mathcal{C} , denoted $[A \rightarrow B] \in \text{Ob}(\mathcal{C})$, and the operation $[_ \rightarrow _] : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}$ is a functor (the *internal hom* functor) that satisfies various natural coherence conditions [MLP77].

This notion of closure describes the properties of implication, with no additional assumptions or connectives. It models logics where the only connective is an implication (such as [GM81]), and under the Curry-Howard isomorphism [SU98], it is used to model purely applicative structures such as combinatory logic — see [LS86] for a good overview. However, it is more common to consider *monoidal closed*, rather than simply *closed* categories. In a monoidal closed category, the internal hom $[_ \rightarrow _] : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}$ is related to the *monoidal tensor* $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ by the adjunction

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B, [A \rightarrow C])$$

The natural interpretation of the tensor in this setting is thus conjunction, and (up to commutativity of conjunction) there is the immediate logical interpretation of

$$(A \wedge B) \Rightarrow C \quad \text{is equivalent to} \quad B \Rightarrow (A \Rightarrow C)$$

Finally, we observe that the adjunction between the monoidal tensor and the internal hom often means that one may be defined in terms of the other. In [McL88], the fact that Abelian groups form a *closed* category

is used, along with the *adjoint functor theorem* to give a tensor product, and hence a *monoidal closed* category of Abelian groups. A similar technique is used to demonstrate the existence of a tensor product for Partial Commutative Monoids in [HS08]. In [MLP77], it is also shown that closed categories may always be embedded into monoidal closed categories.

Thus, from a categorical point of view, there is a strong connection between the two distinct controversial areas of quantum logic – the *tensor product* and the *implication*.

Relating order-theoretic and categorical approaches to logic

Although we have presented order theory and category theory as two competing, opposed, approaches to logic, the reality is more subtle. Relating order-theoretic and category-theoretic approaches has been a very fruitful area of study and has shed light on order theory, category theory, logic and theoretical computing. We refer to [BL73, SA91, LS86] for several examples of a large field.

1.7.3 Compact closed categories

Compact closed categories are symmetric monoidal closed categories where the categorical closure is of a particularly well-behaved form. We refer to [KL80] for details on compact closed categories, including a coherence theorem, [SA96] and [JSV96] for their construction from traced monoidal categories, [SA96, PH97, AHS02, PH04] for logical and computational interpretations, and [PH99] for one-object, or untyped, compact closed categories.

Definition 1.7.1 compact closed categories, dual on objects, dual on arrows

Let \mathbf{C}, \otimes be a symmetric monoidal category, with unit object I . The category \mathbf{C} is called **compact closed** when, for every object $A \in \text{Ob}(\mathbf{C})$, there exists a **dual object** A^* , together with distinguished arrows

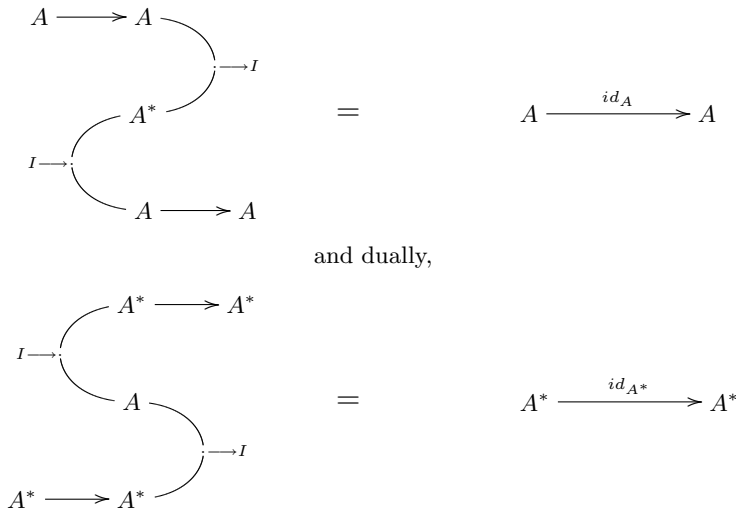
- The unit arrow $\epsilon_A : A \otimes A^* \rightarrow I$
- The counit arrow $\eta_A : I \rightarrow A^* \otimes A$

that satisfy

$$(\epsilon_A \otimes 1_A)(1_A \otimes \eta_A) = 1_A \quad \text{and dually,} \quad (1_{A^*} \otimes \epsilon_A)(\eta_A \otimes 1_{A^*}) = 1_{A^*}$$

Using the diagrammatic notation introduced in [JS91, JSV96], this may be drawn as in Figure 1.3.

Fig. 1.3. Axioms for compact closure

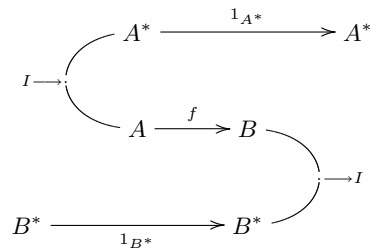


The dual operation on objects $(_)^*$, together with the unit and counit arrows may be used to define the **dual on arrows**. Given $f \in \mathcal{C}(A, B)$, then $f^* \in \mathcal{C}(B^*, A^*)$ is defined by

$$f^* = (1_{A^*} \otimes \epsilon_B)(1_{A^*} \otimes f \otimes 1_{B^*})(\eta_A \otimes 1_{B^*}) : B^* \rightarrow A^*$$

Diagrammatically, this is as shown in Figure 1.4.

Fig. 1.4. The dual operation on arrows



The internal hom is particularly easy to define, being given by $[X \rightarrow Y] = X^* \otimes Y \in \text{Ob}(\mathcal{C})$. The details of the adjunction are then straight-

the identity, scaled by the appropriate renormalisation required for the post-selection of measurement outcome.

1.7.5 Logical interpretations

Although the foundations for quantum mechanics of Section 1.7.4 above are presented categorically, rather than logically, the fact that a monoidal closed (in fact, compact closed) category is key to this system leaves it open to a logical interpretation. An explicit logical interpretation is given in [RD04, AD06, RD07], by analogy with the connectives and structure of linear logic.

The application of compact closure to logical systems arose from analyses of Girard's Geometry of Interaction system [JYG87b, JYG90], a representation of Linear Logic [JYG87a]. We refer to [BCS00] for a good historical overview. The categorical interpretation of Linear Logic as a whole is in terms of $*$ -autonomous categories. However, the Geometry of Interaction system gave a representation of a restricted fragment (the *multiplicatives*). It was also degenerate in many ways, including the identification of the conjunction and disjunction (a case is made in [AHS02] that the correct interpretation of the geometry of interaction system is as a *combinatory logic*).

The logical interpretation of the categorical analysis of Section 1.7.4 above is even more degenerate, in that all objects are *self-dual*. In $*$ -autonomous and compact closed categories, the dual $*$ is interpreted as the logical negation. However, there remains a monoidal tensor and (via the monoidal closure) an internal hom functor that satisfy the required properties for a logical system.

We refer to [RD04, AD06] for more details.

1.8 Compact closure, teleportation, and partial isometries

The category of partial isometries can reasonably be considered as a categorification of von Neumann - Birkhoff quantum logic. However, from Section 1.6.1 and the interpretation of Lemma 1.4.2, we do not expect to be able to describe all quantum operations within the category of partial isometries.

In order to consider whether the category of partial isometries, as a categorification of von Neumann-Birkhoff quantum logic, fits into the general 'categorical foundations' framework, we consider whether

a (post-selected version of) the teleportation protocol [BBJC93] can be expressed in the category $(\mathbf{pIsom}_{FD}, \otimes)$.

1.8.1 Is $(\mathbf{pIsom}_{FD}, \otimes)$ compact closed ?

Before answering the above question, we need to explain why (apart from wishful thinking) we might think that $(\mathbf{pIsom}_{FD}, \otimes)$ should be compact closed – at least, in the finite-dimensional case. A suggestive, but incorrect, train of thought is as follows:

Non-theorem 1.8.1 *The category $(\mathbf{pIsom}_{FD}, \otimes)$ is compact closed, with self-dual objects.*

NON-PROOF. Consider the defining identity of compact closure (in the self-dual case):

$$\lambda(\epsilon_A \otimes 1_A)(1_A \otimes \eta_A)\rho^{-1} = 1_A$$

If, as in the categorical foundations described in Section 1.7.4, we interpret the counit η_H and unit ϵ_H as the bra and ket $|\mathcal{B}ell\rangle : \mathbb{C} \rightarrow H$ and $\langle \mathcal{B}ell| : H \rightarrow \mathbb{C}$, we observe that these are partial isometries. Hence, as partial isometries are closed under the tensor product, $(\epsilon_A \otimes 1_A)$ and $(1_A \otimes \eta_A)$ are both partial isometries. The identity isomorphisms $\rho^{-1} : H \cong H \otimes \mathbb{C}$ and $\lambda : \mathbb{C} \otimes H \cong H$ are trivially unitary, and so (by Corollary 1.3.3) the linear maps

$$\mathcal{T}ele = \lambda(\epsilon_A \otimes 1_A) \quad \text{and} \quad \mathcal{P}ort = (1_A \otimes \eta_A)\rho^{-1}$$

are both partial isometries.

Based on the categorical foundations, we wish to claim that the composite $\mathcal{T}ele\mathcal{P}ort : H \rightarrow H$ is the identity map (again, a partial isometry). From Theorem 1.3.2, the composite KL of two partial isometries is itself a partial isometry exactly when the final projector of L commutes with the initial projector of K , in which case (by Lemma 1.5.3) $K \circ L = KL$.

We thus wish to conclude that the initial projector of $\mathcal{T}ele$ commutes with the final projector of $\mathcal{P}ort$, and so

$$\mathcal{T}ele \circ \mathcal{P}ort = \mathcal{T}ele\mathcal{P}ort = 1_H$$

◇

To see that this reasoning is incorrect, we explicitly exhibit the final projector of $\mathcal{P}ort$, and the initial projector of $\mathcal{T}ele$, and show that these do not commute.

Lemma 1.8.2 Consider the partial isometries $\mathcal{T}ele$ and $\mathcal{P}ort$, defined in Non-Theorem 1.8.1 above. Then the initial and final projectors $E_{\mathcal{T}ele}$ and $F_{\mathcal{P}ort}$ do not commute, so $\mathcal{T}ele \circ \mathcal{P}ort \neq \mathcal{T}ele\mathcal{P}ort$.

Proof Consider a complex N -dimensional space H , with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$. The maximally entangled Bell state is given by $\mathcal{B}ell = \frac{1}{\sqrt{N}} \sum_{j=1}^N \mathbf{e}_i \otimes \mathbf{e}_i$, so the unit and counit maps are

$$\frac{1}{\sqrt{N}} \sum_{j=1}^N |\mathbf{e}_j \mathbf{e}_j\rangle \quad \text{and} \quad \frac{1}{\sqrt{N}} \sum_{k=1}^N \langle \mathbf{e}_k \mathbf{e}_k |$$

respectively. From this, the partial isometries $\mathcal{T}ele : H \otimes H \otimes H \rightarrow H$ and $\mathcal{P}ort : H \rightarrow H \otimes H \otimes H$ may be given explicitly by

$$\mathcal{T}ele = \frac{1}{\sqrt{N}} \sum_{a,b=1}^N |\mathbf{e}_a\rangle \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a | \quad \text{and} \quad \mathcal{P}ort = \frac{1}{\sqrt{N}} \sum_{i,j=1}^N |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle \langle \mathbf{e}_i |$$

Thus, the final projector of $\mathcal{P}ort$ may be given by

$$\begin{aligned} F_{\mathcal{P}ort} &= \left(\frac{1}{\sqrt{N}} \sum_{i,j=1}^N |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle \langle \mathbf{e}_i | \right) \left(\frac{1}{\sqrt{N}} \sum_{k,l=1}^N |\mathbf{e}_k \mathbf{e}_l \mathbf{e}_l\rangle \langle \mathbf{e}_k | \right)^* \\ &= \frac{1}{N} \sum_{i,j,k,l=1}^N |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle \delta_{ik} \langle \mathbf{e}_k \mathbf{e}_l \mathbf{e}_l | = \frac{1}{N} \sum_{i,j,l=1}^N |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle \langle \mathbf{e}_i \mathbf{e}_l \mathbf{e}_l | \end{aligned}$$

Similarly, the initial projector of $\mathcal{T}ele$ is given by

$$\begin{aligned} E_{\mathcal{T}ele} &= \left(\frac{1}{\sqrt{N}} \sum_{c,d=1}^N |\mathbf{e}_d \mathbf{e}_d \mathbf{e}_c\rangle \langle \mathbf{e}_c | \right) \left(\frac{1}{\sqrt{N}} \sum_{a,b=1}^N |\mathbf{e}_a\rangle \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a | \right) \\ &= \frac{1}{N} \sum_{a,b,c,d=1}^N |\mathbf{e}_d \mathbf{e}_d \mathbf{e}_c\rangle \delta_{ac} \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a | = \frac{1}{N} \sum_{a,b,d=1}^N |\mathbf{e}_d \mathbf{e}_d \mathbf{e}_a\rangle \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a | \end{aligned}$$

Direct calculation verifies that $E_{\mathcal{T}ele} F_{\mathcal{P}ort} \neq F_{\mathcal{P}ort} E_{\mathcal{T}ele}$ and so by Proposition 1.3.2, $\mathcal{T}ele \circ \mathcal{P}ort \neq \mathcal{T}ele\mathcal{P}ort$. \square

What, then has gone wrong in the reasoning in Non-Theorem 1.8.1? Our claim is that the renormalisation, or implicit post-selection, in Section 1.7.4 is incompatible with a study of teleportation via partial isometries, as follows:

Theorem 1.8.3 Consider the partial isometries $\mathcal{T}ele : H \rightarrow H \otimes H \otimes H$ and $\mathcal{P}ort : H \otimes H \otimes H \rightarrow H$ of Non-Theorem 1.8.1. Their composite, as linear maps, is $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N}1_H$ and therefore their composite in the category \mathbf{pIsom}_{FD} is $\mathcal{T}ele \circ \mathcal{P}ort = 0_H$.

Proof By direct calculation,

$$\begin{aligned} \mathcal{T}ele\mathcal{P}ort &= \frac{1}{N} \sum_{a,b,i,j=1}^N |\mathbf{e}_a\rangle \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a | \mathbf{e}_i \mathbf{e}_j \mathbf{e}_j \rangle \langle \mathbf{e}_i | \\ &= \frac{1}{N} \sum_{a,b,i,j=1}^N |\mathbf{e}_a\rangle \delta_{bi} \delta_{bj} \delta_{aj} \langle \mathbf{e}_i | = \frac{1}{N} \sum_{a=1}^N |\mathbf{e}_a\rangle \langle \mathbf{e}_a | = \frac{1}{N} Id_H \end{aligned}$$

Using the original definition of composition in \mathbf{pIsom} ,

$$\mathcal{T}ele \circ \mathcal{P}ort = \lim_{K \rightarrow \infty} [(\mathcal{T}ele\mathcal{P}ort)(\mathcal{T}ele\mathcal{P}ort)^*]^K \mathcal{T}ele\mathcal{P}ort$$

As $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N}Id_H$ is self-adjoint,

$$\mathcal{T}ele \circ \mathcal{P}ort = \lim_{K \rightarrow \infty} \left[\frac{1}{N^2} Id_H \right]^K \frac{1}{N} Id_H = 0_H$$

□

Why teleportation cannot be expressed in $(\mathbf{pIsom}_{FD}, \otimes)$

A direct calculation has given that, as a straightforward composite of linear maps, $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N}Id_H$. This extraneous factor of $\frac{1}{N}$ occurs simply because the probability of observing this particular experimental outcome (i.e. the Bell state $\mathcal{B}ell = \frac{1}{\sqrt{N}} \sum_{j=1}^N \mathbf{e}_j \otimes \mathbf{e}_j$) is exactly $\frac{1}{N}$.

In the categorical foundations approach, post-selection of measurement outcomes is used to eliminate this scaling factor, and hence give compact closure. In \mathbf{pIsom} , we cannot simply introduce a scaling factor of N at some point to account for post-selection; all partial isometries (except the zero map) have operator norm of 1, and such a ‘scaled’ unit / co-unit pair would no longer be members of the category.

An alternative approach is to work with rays (i.e. one-dimensional complex subspaces) rather than norm-1 vectors of a Hilbert space. In this way, we eliminate any notion of magnitude, and simply leave some generalised notion of angle. However, regardless of scaling, the initial space of $\mathcal{P}ort$ and the final space of $\mathcal{T}ele$ are both 1-dimensional subspaces of $H \otimes H \otimes H$. These subspaces are neither orthogonal, nor co-incident, and their only commonality is the zero-dimensional subspace

$\{0\}$, as shown in Figure 1.1.

1.9 Is $(\mathbf{pIsom}_{FD}, \otimes)$ closed?

So far, we have *not* proved that $(\mathbf{pIsom}_{FD}, \otimes)$ is not compact closed. Rather we have shown that the usual way of demonstrating compact closure (as post-selected teleportation) does not hold in $(\mathbf{pIsom}_{FD}, \otimes)$. We now demonstrate that $(\mathbf{pIsom}_{FD}, \otimes)$ cannot be closed at all (and therefore is certainly not *compact* closed).

To demonstrate that $(\mathbf{pIsom}_{FD}, \otimes)$ is not monoidal closed, we rely on some more in-depth category theory. We first provide some basic definitions, based on those given in [McL88]:

Definition 1.9.1 Diagrams, cones, limits, continuous functors

Let \mathbf{J} be a small, or finite, category. A **diagram** of type \mathbf{J} (or with **index \mathbf{J}**) in a category \mathbf{C} is simply a functor $\Gamma : \mathbf{J} \rightarrow \mathbf{C}$. A **cone to Γ** is then an object $N \in \text{Ob}(\mathbf{C})$, together with a family of arrows of \mathbf{C} indexed by $\text{Ob}(\mathbf{J})$

$$\{\Phi_X : N \rightarrow \Gamma(X)\}_{X \in \text{Ob}(\mathbf{J})}$$

such that the following diagram commutes, for all $f \in \mathbf{J}(X, Y)$:

$$\begin{array}{ccc} \Gamma(X) & \xrightarrow{\Gamma(f)} & \Gamma(Y) \\ & \swarrow \Phi_X & \searrow \Phi_Y \\ & N & \end{array}$$

A **limit** for the diagram $\Gamma : \mathbf{J} \rightarrow \mathbf{C}$ is then a cone $(L, \Lambda_{_})$ to Γ such that every other cone to Γ , $(N, \Phi_{_})$ factors through $(L, \Lambda_{_})$ via a unique arrow $u : N \rightarrow L$ — that is, there exists a unique $u : N \rightarrow L$ that makes the following diagram commute

$$\begin{array}{ccc} & N & \\ & \downarrow u & \\ \Phi_X & L & \Phi_Y \\ & \swarrow \Lambda_X \quad \searrow \Lambda_Y & \\ \Gamma(X) & \xrightarrow{\Gamma(f)} & \Gamma(Y) \end{array}$$

for arbitrary $f \in \mathbf{J}(X, Y)$.

A functor $\Delta : C \rightarrow D$ is called **continuous** if it preserves limits. This is an important property satisfied by many categorically natural functors, including functors that form an adjoint pair.

We now use the notion of continuity to demonstrate that the category of partial isometries cannot be monoidal closed. This demonstration is based on a categorical interpretation of existing criticisms of implication in quantum logic — although it is not restricted to the Sasaki hook, or indeed any of the 5 proposed implications — so we simply sketch an outline, rather than providing full details.

Theorem 1.9.2 *The category $(\mathbf{pIsom}_{FD}, \otimes)$ is not monoidal closed.*

Proof (Sketch)

In a monoidal closed category \mathbf{C}, \otimes , the functor $_ \otimes B : \mathbf{C} \rightarrow \mathbf{C}$ is adjoint to the internal hom. functor $[B, _] : \mathbf{C} \rightarrow \mathbf{C}$, and thus — as a straightforward property of adjoint functors — must be *continuous*.

Now recall that any partially ordered set (P, \leq) can be treated as a category whose objects are the elements of the poset, where there exists a unique arrow from x to y exactly when $x \leq y$. We thus use simple posets as the index category in defining diagrams within \mathbf{pIsom}_{FD} . It is then a straightforward exercise to show that within \mathbf{pIsom}_{FD} , both meets and joins of idempotents are examples of categorical limits. Thus, if \mathbf{pIsom}_{FD} is to be monoidal closed, we require that the functor $_ \otimes H : \mathbf{pIsom}_{FD} \rightarrow \mathbf{pIsom}_{FD}$ preserves meets and joins — and indeed, such limits generally.

The interaction of the tensor product with the orthomodular lattice structure has been well-studied within quantum logic — we refer to [TW04] for the result that the tensor product of an orthomodular lattice L with an arbitrary lattice L' is itself an orthomodular lattice exactly when L' is a Boolean lattice.

Thus continuity implies the distributive law; the point at which limits fail is exactly the point at which the classical and quantum-mechanical order theories differ, and so we deduce that $(\mathbf{pIsom}_{FD}, \otimes)$ cannot be monoidal closed. \square

The adjoint functor theorem, and orthomodularity

The usual method of proving or disproving that a complete category is monoidal closed is via an application of Freyd's *adjoint functor theorem* [McL88] to the description of categorical closure as an adjunction. It is

sometimes claimed that this theorem can be used to demonstrate that no closed structure can be built upon orthomodular lattices.

However, it is also possible to use the adjoint functor theorem to demonstrate that the category of Hilbert spaces and linear maps is not closed. Recall that it is only the category of *finite-dimensional* Hilbert spaces that is compact closed [ABP98]. When working in the finite-dimensional case, the resulting categories (whether based on linear maps, partial isometries, or orthomodular lattice morphisms), are no longer complete categories — and therefore the adjoint functor theorem is inapplicable.

1.10 Future Directions

We have established that, far from being the ‘monstrosities’ of [JYG87b], partial isometries do in fact have a rich categorical structure, closely related to Birkhoff/von Neumann quantum logic. This categorical structure has also let us see that there does not appear to be a natural overlap between the ‘categorical foundations’ and ‘orthomodular lattices’ approaches to foundations of quantum mechanics. Although this may be unsurprising to researchers in both fields, hopefully this paper has also provided a new perspective on why these foundational approaches appear to be incompatible.

There remain several unanswered questions. One of the most pressing is: what sort of partially ordered structure is provided by the Halmos-McLaughlin partial order? Although this paper has concentrated on the category theory of partial isometries, we will demonstrate in a future paper that $(\mathbf{pIsom}(H, K), \leq)$ is in fact a complete partial order (CPO), where the down-closure of each maximal element is an orthomodular lattice. Whether or not it has more structure (e.g. is it a Scott domain) is at present an open question.

There also remains to consider the monoidal structure of \mathbf{pIsom} , with respect to the direct sum, rather than the tensor product. This is closely related to both the partial ordering, and the inverse category structure. It is demonstrated in [PH07] that all inverse categories satisfying very light requirements (including \mathbf{pIsom}) have a categorical trace. However, this is a trace on the direct sum structure, rather than the tensor product structure, so does not have the usual natural interpretation in terms of composite systems.

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