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Categorical analogues of monoid semirings

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This paper introduces and studies a categorical analogue of the familiar monoid semiring construction. We demonstrate that the algebraic construction generalises in a natural way to certain categories with additional structure based on, but generalising, that found in the field of algebraic program semantics. This allows us to set up categorical structure required to define the discrete Fourier transform of a category, as a generalisation of the discrete Fourier transform of a group-ring. A straightforward example is given, as a prototype for a more general theory.

1. Introduction

Discrete Fourier transforms play a significant role in both quantum mechanics generally, and in quantum algorithms believed to exhibit significant complexity gains compared to their classical counterparts. In (Nicholson 1971), a *discrete Fourier transform (DFT)* is simply defined in terms of an isomorphism from a group ring to a direct sum of rings. Motivated by Nicholson's elegant algebraic definition, this paper aims to place the group-ring construction (or rather, the more general monoid-semiring construction) within a categorical setting — in particular, a setting based on formal axiomatisations of algebraic program semantics. To this end, we demonstrate that the monoid-semiring construction has a natural category-theoretic generalisation that we call the Cauchy product. This categorical framework then allows us to give a functorial analogue of Nicholson's algebraic description of DFTs. We also consider conditions on categories that imply the existence of such functors.

1.1. *Cauchy products and monoid semirings*

In the formal theory of power series, an infinite power series over some complex variable z, given as $P = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots$ may be treated as simply a function $P : \mathbb{N} \to \mathbb{C}$. Given

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another formal power series $Q : \mathbb{N} \to \mathbb{C}$ over the same variable, their *Cauchy product* is the formal power series $(Q * P) : \mathbb{N} \to \mathbb{C}$ given by $(Q * P)(n) = \sum_{n=y+x} q(y)p(x)$.

A formal power series $p : \mathbb{N} \to \mathbb{C}$ absolutely converges within the unit disk $\{\|z\| \leq 1\}$ when the sum $\sum_{n\in\mathbb{N}} p(n)$ converges absolutely and it is a straightforward result of analysis (Titchmarsh 1983) that the Cauchy product of two formal power series that absolutely converge within the unit disk itself absolutely converges within the unit disk (and much more general conditions may also lead to convergence — we again refer to (Titchmarsh 1983)).

When restricting formal power series to the case where only a finite number of coefficients are non-zero, convergence is guaranteed not only within the unit disk, but for all $z \in \mathbb{C}$. Algebraically, this naturally generalises to the familiar theory of monoid semirings (Golan 1999):

Definition 1.1. *Monoid semirings*
Let $(M, \cdot)$ be a monoid, and $(R, \times, +)$ be a unital semi-ring. The **monoid semiring** $R[M]$ is the unital semiring whose elements are functions $\eta : M \to R$ such that $\|\{\eta(m) \neq 0\}_{m\in M}\| < \infty$. The multiplication and addition in this semiring are given by

— $(\eta \times \mu)(m) = \sum_{m=qp} \eta(q)\mu(p)$,
— $(\eta + \mu)(m) = \eta(m) + \mu(m)$.

The additive identity is the function $0(m) = 0_R \ \forall m \in M$, and the multiplicative identity is the function $1(m) = \begin{cases} 1_R & m = 1_M \\ 0_R & \text{otherwise.} \end{cases}$ When $(M, \cdot)$ is a group, $R[M]$ is called the **group semiring**; similarly, when R is a ring, then $R[M]$ is called the **monoid (or group) ring**.

This paper generalises the above construction of monoid semirings in two ways:

— The finite sums of a semiring are replaced by a more general axiomatic summation of (possibly infinite) indexed families.
— The monoids $(M, \cdot)$ and $(R, \times)$ are replaced by categories. Thus, the unital ring R is replaced by a category with some appropriate notion of summation on homsets.

We then demonstrate how the algebraic description of DFTs given by (Nicholson 1971) has a functorial analogue within this more general framework.

2. An axiomatic notion of summation

For the program outlined above, we replace semirings with categories equipped with a partial summation on hom-sets. The overall intention of this paper is to place the monoid semiring construction, and the discrete Fourier transform within the setting of algebraic program semantics. However (as discussed in the Appendix, Section 10) axiomatisations of summation commonly used within algebraic program semantics have properties that rule out the very analytic notions of summation needed for the classical theory of DFTs.

We therefore introduce a very general axiomatisation of summation that includes, as special cases, various notions of (partial, indexed) summation from both theoretical computer science and analysis. By comparison with other notions of summation discussed in the Appendix (Section 10), this is a very weak axiomatisation — in particular, the expressive power we require comes from both the axioms we now present, and the axioms for the interaction of summation and composition given in Section 3.

Definition 2.1. *Partial Commutative Monoids (PCMs)*
Given sets M and I, an **I-indexed family of elements** of M is defined to be a function $x : I \to X$. For simplicity, we denote this by $\{x_i\}_{i\in I}$. When I is countable (i.e. either finite or denumerably infinite), we say that this is a **countably indexed family**.

A **partial commutative monoid** or **PCM** is a non-empty set M together with a partial function Σ from countably indexed families of M to elements of M. An indexed family of elements of M is called **summable** when it is in the domain of Σ, and summation is required to satisfy the following two axioms:

1 The **Unary Sum axiom** Any family $\{x_i\}_{i\in I}$, where $I = \{i'\}$ is a singleton set, is summable, and $\sum_{i\in I} x_i = x_{i'}$.
2 The **Weak Partition-Associativity axiom** Let $\{x_i\}_{i\in I}$ be a *summable* family, and let $\{I_j\}_{j\in J}$ be a countable partition† of I. Then $\{x_i\}_{i\in I_j}$ is summable for every $j \in J$, as is $\{\sum_{i\in I_j} x_i\}_{j\in J}$, and

$$\sum_{i\in I} x_i = \sum_{j\in J}\left(\sum_{i\in I_j} x_i\right)$$

Given a summable family $x = \{x_i\}_{i\in I}$, we may write $\Sigma(x)$ (unambiguously) as $\sum_{i\in I} x_i$. In particular, if $I = \{1, \dots, n\}$, we write $\Sigma(x) = x_1 + x_2 + x_3 + \cdots + x_n$, and if $I = \mathbb{N}$, $\Sigma(x) = x_1 + x_2 + x_3 + \cdots$. Notice that by Weak Partition Associativity, we may equate different partitions of a summable family x, for example:

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots &= x_1 + (x_2 + x_3 + \cdots + x_n + \cdots) \\ &= (x_1 + x_2) + (x_3 + x_4) + \cdots + (x_n + x_{n+1}) + \cdots . \end{aligned}$$

In the Appendix (Section 10) we give various examples of PCMs, and compare this formalism to other axiomatisations of summation used in various fields. However, we first demonstrate that the indexed summation of a PCM is well-defined — that is, the partial summation operation preserves "equivalent" families.

Definition 2.2. *Equivalence of indexed families*
Given countably indexed families of elements of some set M, $x : I \to M$ and $y : J \to M$, we say that x and y are **equivalent families**, written $x \cong y$, when there exists a bijection $\varphi : I \to J$ such that $y\circ\varphi = x$. That is, $y_{\varphi(i)} = x_i$ for all $i \in I$. It is immediate that equivalence of families is an equivalence relation.

We now demonstrate that in a PCM, equivalence preserves summability, and equivalent summable families sum to the same element.

† Following (Manes, Arbib 1986), we also allow countably many I_j to be empty.

Proposition 2.3. Suppose $x \cong y$ are two equivalent families of a partial commutative monoid M. If $\Sigma(y)$ is well defined, then so is $\Sigma(x)$ and they are equal. That is, writing $x = \{x_i\}_{i\in I}$, and similarly for y, then $\sum_{j\in J} y_j$ exists exactly when $\sum_{i\in I} x_i$ exists, in which case $\sum_{j\in J} y_j = \sum_{i\in I} x_i$.

Proof. Let $\varphi : I \to J$ be the bijection exhibiting the equivalence of x and y, so $x_i = y_{\varphi(i)}$, for all $i \in I$. Let $J_i = \{\varphi(i)\}$ be a singleton set; hence $\{J_i \mid i \in I\}$ is a partition of J. Then we have:

$$\begin{aligned}\sum_{j\in J} y_j &= \sum_{i\in I}\sum_{j\in J_i} y_j && \text{Weak partition associativity}\\ &= \sum_{i\in I} y_{\varphi(i)} && \text{Defn, and Unary sum axiom}\\ &= \sum_{i\in I} x_i && \text{Defn}\end{aligned}$$

Thus, as equivalence of families is an equivalence relation, the (partial, indexed) summation of a PCM is well-defined. □

The following basic properties of PCMs will be used throughout:

Proposition 2.4. Let (M, Σ) be a PCM. Then

1 *(Summable Subfamilies)* Let $\{x_i\}_{i\in I}$ be a summable family of M. Then any subfamily $\{x_i\}_{i\in K}$, where $K \subseteq I$, is also summable.
2 *(Existence of Zero)* The empty set is summable, and $x + \{\} = x = \{\} + x$ for all $x \in M$. Hence it is a zero for M, and we write $0 = \sum\{\}$.
3 *(Sums of Zeros)* For any index set I, let $0_I : I \to M$ denote the constantly zero family (so $O_I(i) = 0$, for all $i \in I$). Then 0_I is summable, and $\Sigma_I 0_I = 0$. More generally, for any element $x \in M$, $x + 0 + 0 + 0 + \cdots = x$ (where $0 + 0 + \cdots$ denotes (the sum of) either a finite or infinite sequence of 0's).

Proof. The proofs of (1) and (2) below are based on very similar proofs (for the special case of partially additive monoids – see Appendix A) presented in (Manes, Arbib 1986).

1 *(Summable Subfamilies)* Any subset $K \subseteq I$ defines a partition of I, namely $\{K, I\backslash K\}$. By Weak Partition Associativity, $\sum_{i\in K} x_i$ exists.
2 *(Existence of Zero)* As M is by definition non-empty, the unary sum axiom implies that the set of summable families is also non-empty. The empty family is a subfamily of any summable family; hence letting $K = \emptyset$ in the partition above, we see that the empty family $\{\}$ is summable. It is then immediate that $\sum\{\} = 0$ is a zero for the summation operation, and $0 + x = x = x + 0$ exists for arbitrary $x \in M$.
3 *(Sums of Zeros)* Pick any partition of I whose first cell is I itself, and the remaining cells are empty (the number of empty cells is either finite or infinite, depending upon whether one wishes a finite (resp. infinite) sum of 0's. For example, write $I = I_1 \uplus (\uplus_{n>1} I_n)$, where $I_1 = I$, $I_i = \emptyset$, if $i > 1$. If $x = \{x_i\}_{i\in I}$ is an I-indexed *summable* family, then by Weak Partition Associativity we have: $\sum_{i\in I} x_i = \sum_{i\in I_1} x_i +$

$\sum_{n>1}(\sum_{i\in I_n} x_i) = \sum_{i\in I_1} x_i + 0 + 0 + \cdots$. Now pick a singleton family $\{x\}$, so $\Sigma(x) = x$. The result follows.

□

We now define homomorphisms of PCMs, and show that the class of all PCMs, together with this notion of homomorphism, forms a category:

Definition 2.5. *PCM homomorphisms, the category of PCMs*
A **homomorphism** of PCMs is a function $f : (M, \Sigma) \to (N, \Sigma')$ satisfying the following natural property:

Given a summable family $\{m_i\}_{i\in I}$ of (M, Σ), then $\{f(m_i)\}_{i\in I}$ is a summable family of (N, Σ'), and $f\left(\Sigma_{i\in I} m_i\right) = \Sigma'_{i\in I} f(m_i)$.

Proposition 2.6. The class of all PCMs, together with the above notion of homomorphism, forms a category that we denote **PCM**.

Proof. First note that for a PCM (M, Σ^M) the identity function $1_M : M \to M$ is a PCM homomorphism. Next, consider PCM homomorphisms $f : (A, \Sigma^A) \to (B, \Sigma^B)$ and $g : (B, \Sigma^B) \to (C, \Sigma^C)$, together with a summable family $\{a_i\}_{i\in I}$. Then the function $gf : A \to C$ satisfies $g\left(f\left(\Sigma^A_{i\in I} a_i\right)\right) = g\left(\Sigma^B_{i\in I} f(a_i)\right) = \Sigma^C_{i\in I} gf(a_i)$. (Note these sums are required to exist, by the definition of PCM homomorphism). Thus gf is a PCM homomorphism from (A, Σ^A) to (C, Σ^C). Finally, associativity of composition follows from the associativity of composition for functions. □

Examples of PCMs are given in the Appendix (Section 10). For the program outlined in Section 1, we now require categories whose hom-sets are PCMs, together with a specified interaction between summation and composition.

3. Categories with a notion of summation on hom-sets

We now introduce a certain class of categories whose hom-sets are PCMs, together with axioms for the interaction of summation and composition:

Definition 3.1. *PCM-categories*
We define a **PCM-category** to be a locally small[‡] category $\mathcal{C}$, together with, for all $X, Y \in Ob(\mathcal{C})$, a partial function $\Sigma^{(X,Y)}$ from countably indexed families over $\mathcal{C}(X, Y)$ to $\mathcal{C}(X, Y)$ (we will often omit the superscript, when this is clear from the context).

This class of partial functions is required to satisfy the following axioms:

1 **(PCM-structure on hom-sets)**
$\left(\mathcal{C}(X, Y), \Sigma^{(X,Y)}\right)$ is a PCM, for all $X, Y \in Ob(\mathcal{C})$.

[‡] i.e. we allow for a proper class of objects, but require that all homsets are indeed sets.

2 **(Strong distributivity)**
Given summable families $\{f_i \in \mathcal{C}(X,Y)\}_{i\in I}$ and $\{g_j \in \mathcal{C}(Y,Z)\}$, then $\{g_j f_i \in \mathcal{C}(X,Z)\}_{(j,i)\in J\times I}$ is a summable family satisfying

$$\sum_{(j,i)\in J\times I}^{(X,Z)} g_j f_i = \left(\sum_{j\in J}^{(Y,Z)} g_j \right) \left(\sum_{i\in I}^{(X,Y)} f_i \right)$$

We consider examples of PCM-categories in the Appendix (Section 10), and properties of PCM-categories in Section 3.1 below.

PCM-categories, and categorical enrichment A very natural question at this point is whether a "PCM-category" is a category enriched over some suitable (monoidal, or closed) category of PCMs. We refer to Section 9 for this question.

3.1. *Properties of PCM-categories*

As may be expected, the strong distributivity property, together with the unary sum axiom, implies the usual left- and right- distributivity laws:

Proposition 3.2. Let $\left(\mathcal{C}, \Sigma^{(\_,\_)}\right)$ be a PCM category, and let $\{g_i \in \mathcal{C}(Y,Z)\}_{i\in I}$ be a summable family. Then, for all arrows $f \in \mathcal{C}(X,Y)$ and $h \in \mathcal{C}(Z,T)$,

$$\{hg_i \in \mathcal{C}(Y,T)\}_{i\in I} \quad \text{and} \quad \{g_i f \in \mathcal{C}(X,Z)\}_{i\in I}$$

are summable families, and

$$h\left(\sum_{i\in I} g_i\right) = \sum_{i\in I}(hg_i) \quad \text{and} \quad \left(\sum_{i\in I} g_i\right) f = \sum_{i\in I}(g_i f)$$

Proof. Consider the index set $A = \{a'\}$, and the indexed family $\{h_a\}_{a\in A}$ given by $h_{a'} = h$. By the unitary sum axiom $h = \sum_{a\in A} h_a$, and so

$$h\sum_{i\in I} g_i = \left(\sum_{a\in A} h_a\right)\left(\sum_{j\in J} g_j\right)$$

By strong distributivity

$$\left(\sum_{a\in A} h_a\right)\left(\sum_{j\in J} g_j\right) = \sum_{(a,j)\in A\times J} h_a g_j$$

As A is a single element set, $A \times J \cong J$, and $h_a = h$. Therefore, by Proposition 2.3,

$$h\sum_{j\in J} g_j = \sum_{j\in J} hg_j$$

The proof for the opposite distributive law is almost identical. □

Corollary 3.3. Every PCM-category has zero arrows.

Proof. In a PCM-Cat. $\left(\mathcal{C}, \Sigma^{(\_,\_)}\right)$ we define, for all $X, Y \in Ob(\mathcal{C})$, $0_{XY} = \sum\{\} \subseteq \mathcal{C}(X,Y)$. Then by the above distributive laws, for all $f \in \mathcal{C}(Y,Z)$, $f0_{XY} = \sum\{\} \subseteq \mathcal{C}(X,Z)$, and hence $f_{XY} = 0_{XZ}$. Similarly $0_{XY}g = 0_{WY}$, for all $g \in \mathcal{C}(W,X)$. □

The usual treatment of distributivity The usual approach in algebraic program semantics is to take the above left- and right- distributivity laws as axiomatic, and use the (much stronger) notion of summation to prove an analogue of strong distributivity. This is described in the Appendix (Section 10). We do not take this approach, because the axiomatisation of summation this requires is too strong for our purposes — it imposes the *positivity property* that $x+y=0 \ \Rightarrow\ x=0=y$, ruling out exactly the structures we will require in order to consider analogues of the discrete Fourier transform. From the examples given in the Appendix, the PCM axiomatisation does not imply the positivity property.

We now consider some implications of strong distributivity:

Proposition 3.4. Let $\left(\mathcal{C}, \Sigma^{(\_,\_)}\right)$ be a PCM-cat. and let $\{g_j \in \mathcal{C}(Y,Z)\}_{j\in J}$ and $\{f_i \in \mathcal{C}(X,Y)\}_{i\in I}$ be summable families. Then $\sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right)$ and $\sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right)$ are both defined, and

$$\sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right) = \sum_{(i,j)\in I\times J} g_j f_i = \sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right)$$

Proof. By the strong distributivity property, the family $\{g_j f_i \in \mathcal{C}(X,Z)\}_{(j,i)\in J\times I}$ is summable. Now consider the partition of $J \times I$ given by $\{\{(j,i)\}_{j\in J}\}_{i\in I}$. By the weak partition-associativity axiom, for arbitrary fixed $i \in I$ the family $\{g_j f_i\}_{j\in J}$ is summable, as is $\left\{\sum_{j\in J} g_j f_i\right\}_{i\in I}$ and

$$\sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right) = \sum_{(i,j)\in I\times J} g_j f_i$$

The dual identity

$$\sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right) = \sum_{(i,j)\in I\times J} g_j f_i$$

follows by partitioning $J \times I$ as $\{\{(j,i)\}_{i\in I}\}_{j\in J}$, and therefore

$$\sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right) = \sum_{(i,j)\in I\times J} g_j f_i = \sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right)$$

□

Proposition 3.5. Let $\left(\mathcal{C}, \Sigma^{(\_,\_)}\right)$ be a PCM-cat. and let $\{s_i \in \mathcal{C}(X,X)\}_{i\in I}$ be a summable family. Then for all $n > 0$, the family

$$\{s_{i_n} s_{i_{n-1}} \ldots s_{i_2} s_{i_1} \in \mathcal{C}(X,X)\}_{(i_n,\ldots i_1)\in I^n}$$

is summable, as is all its subfamilies.

Proof. *(By induction)* The result is trivially true for $n = 1$. Now assume it holds for some $k > 0$. Then by strong distributivity,

$$\{s_i s_{i_k} \dots s_{i_1} \in \mathcal{C}(X,X)\}_{(i,(i_n,\dots i_1))\in I\times I^k}$$

is also summable, and our result follows by induction. Finally, recall the summable subfamilies property (Proposition 2.4). □

Corollary 3.6. Let $(\mathcal{C}, \Sigma^{(\_,\_)})$ be a PCM-cat. and let $F = \{f_i \in \mathcal{C}(X,X)\}_{i\in I}$ be a summable family containing the identity. Then

1 Arbitrary finite subsets of the submonoid of $\mathcal{C}(X,X)$ generated by F are summable.
2 Let F' denote the indexed subfamily given by removing all occurrences of 1_X from F. When there exists some word w in the subsemigroup generated by F' satisfying $w = 1_X$, then
   (a) The sum $\sum_{i=1}^{M} 1_X$ exists, for all $M \in \mathbb{N}$.
   (b) For all $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(W,X)$, the sums

$$\sum_{i=1}^{M} f \in \mathcal{C}(X,Y) \quad \text{and} \quad \sum_{i=1}^{M} g \in \mathcal{C}(W,X)$$

   exist, for all $M \in \mathbb{N}$.

Proof.

1 Consider a finite subset $T \subseteq F^* \subseteq \mathcal{C}(X,X)$. As T is finite, there exists some $K \in \mathbb{N}$ such that each $t \in T$ may be written as a distinct word of no more than K members of F. However, since F contains the identity, each word of T may be written as a distinct word of exactly K members of $\{f_i\}_{i\in I}$. Thus, our result follows by Proposition 3.5 above and the summable subfamilies property (Proposition 2.4).
2 We now assume the additional condition on F given above:
   (a) Let us write $w = 1_X$ as a word of K elements of F'. Then by Proposition 3.5 above, the family

$$\{1_X^{K(M-N)} w 1_X^{KN}\}_{N=1..M}$$

   is summable. However, $1_X^{K(M-N)} w 1_X^{KN} = 1_X$, for all $N = 1..M$. Therefore $\sum_{N=1}^{M} 1_X$ exists, as does $\sum_{N=1}^{M'}$, for all $0 < M' < M$, by the summable subfamilies property (Proposition 2.4).
   (b) By distributivity (Proposition 3.2 above) $\{f 1_X \in \mathcal{C}(X,Y)\}_{1=1..M}$ exists, and hence $\sum_{i=1}^{M} f$ exists. The proof for arbitrary $g \in \mathcal{C}(W,X)$ is similar.

□

3.2. *The category of PCM-categories*

The class of all PCM-categories is itself a category:

Definition 3.7. *PCM-functors, the category* $\mathbf{Cat}_\Sigma$
Given PCM-categories $\mathcal{C}, \mathcal{D}$, we say that a functor $\Gamma : \mathcal{C} \to \mathcal{D}$ is a **PCM-functor** when:

— Given a summable family $\{f_i \in \mathcal{C}(X, Y)\}_{i \in I}$, then $\{\Gamma(f_i) \in \mathcal{D}(\Gamma(X), \Gamma(Y))\}$ is a summable family, and

$$\Gamma\left(\sum_{i \in I} f_i\right) = \sum_{i \in I} \Gamma(f_i)$$

We denote the category of all PCM-categories and PCM-functors by $\mathbf{Cat}_\Sigma$.

Proposition 3.8. $\mathbf{Cat}_\Sigma$ is well-defined.

Proof. First note that identity functors on PCM-categories are trivially PCM-functors. To prove compositionality, consider two PCM-functors $\Gamma \in \mathbf{Cat}_\Sigma(\mathcal{C}, \mathcal{D})$ and $\Delta \in \mathbf{Cat}_\Sigma(\mathcal{D}, \mathcal{E})$. By definition, for any summable family $\{f_i \in \mathcal{C}(X, Y)\}_{i \in I}$, the family $\{\Gamma(f_i) \in \mathcal{D}(\Gamma(X), \Gamma(Y))\}_{i \in I}$ is summable, as is $\{\Delta\Gamma(f_i) \in \mathcal{E}(\Delta\Gamma(X), \Delta\Gamma(Y))\}_{i \in I}$. Then, also by definition of PCM-functors,

$$\Delta\left(\Gamma\left(\sum_{i \in I} f_i\right)\right) \;=\; \Delta\left(\sum_{i \in I} \Gamma(f_i)\right) = \sum_{i \in I} \Delta\Gamma(f_i)$$

and hence $\Delta\Gamma$ is also a PCM-functor. Finally, associativity follows from the usual associative property for functors, and thus $\mathbf{Cat}_\Sigma$ is well-defined. □

We also have finite products of PCM-categories:

Proposition 3.9. The category $\mathbf{Cat}_\Sigma$ has finite products.

Proof. Consider $\mathcal{C}, \mathcal{D} \in Ob(\mathbf{Cat}_\Sigma)$. We define their product $\mathcal{C} \times \mathcal{D}$ in a similar way to the usual product of categories: objects are pairs (A, X), where $A \in Ob(\mathcal{C})$ and $X \in Ob(\mathcal{D})$. The homset $(\mathcal{C} \times \mathcal{D})((A, X), (B, Y))$ is exactly the Cartesian product $\mathcal{C}(A, B) \times \mathcal{D}(X, Y)$, with the usual component-wise composition.

It remains to consider summation on homsets. The projections $\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$, and $\pi_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ are defined exactly as in the usual product of categories. For non-empty I, a family $\{f_i \in (\mathcal{C} \times \mathcal{D})((A, X), (B, Y))\}_{i \in I}$ is summable exactly when

$$\{\pi_1(f_i) \in \mathcal{C}(A, B)\}_{i \in I} \;\text{ and }\; \{\pi_2(f_i \in \mathcal{D}(X, Y)\}_{i \in I}$$

are summable, in which case

$$\sum_{i \in I} f_i \;=\; \left(\sum_{i \in I} \pi_1(f_i)\;,\; \sum_{i \in I} \pi_2(f_1)\right) \in (\mathcal{C} \times \mathcal{D})((A, X), (B, Y))$$

When I is empty, we simply take $\sum_{i \in I} f_i = (0_{AB}, 0_{XY})$.

We now demonstrate that this definition satisfies the required universal property for a categorical product: Given PCM-functors $\Gamma_1 \in \mathbf{Cat}_\Sigma(\mathcal{X}, \mathcal{C})$ and $\Gamma_2 \in \mathbf{Cat}_\Sigma(\mathcal{X}, \mathcal{D})$, we define $\langle \Gamma_1, \Gamma_2 \rangle : \mathcal{X} \to \mathcal{C} \times \mathcal{D}$ by

— *On objects* $\langle \Gamma_1, \Gamma_2 \rangle(R) = (\Gamma_1(R), \Gamma_2(R))$, for all $R \in Ob(\mathcal{X})$.
— *On arrows* $\langle \Gamma_1, \Gamma_2 \rangle(f) = (\Gamma_1(f), \Gamma_2(f)) \in (\mathcal{C} \times \mathcal{D})((\Gamma_1(R), \Gamma_2(R)), (\Gamma_1(S), \Gamma_2(S))$, for all $f \in \mathcal{X}(R, S)$.

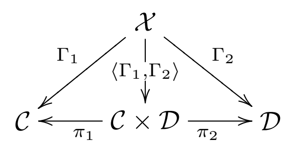
Functoriality of $\langle \Gamma_1, \Gamma_2 \rangle$ is immediate. To demonstrate that it is also a PCM-functor, consider a summable family $\{f_i \in \mathcal{X}(R,S)\}_{i\in I}$. Then

$$\{\langle \Gamma_1, \Gamma_2\rangle (f_i)\}_{i\in I} \;=\; \{(\Gamma_1(f_i), \Gamma_2(f_i))\}_{i\in I}$$

which is summable by definition of summability in $\mathcal{C}\times\mathcal{D}$. By the definition of summation in $\mathcal{C}\times\mathcal{D}$,

$$\sum_{i\in I} \langle \Gamma_1, \Gamma_2\rangle (f_i) \;=\; \left(\sum_{i\in I} \Gamma_1(f_i) \;,\; \sum_{i\in I} \Gamma_2(f_i) \right)$$

and thus $\langle \Gamma_1, \Gamma_2 \rangle$ is also a PCM-functor. Finally, by the usual theory of product categories, the following diagram commutes:



□

4. The categorical Cauchy product

We are now in a position to introduce a categorical analogue of the monoid semiring construction of Definition 1.1. In honour of the original axiomatisation of such products in the theory of formal power series, we refer to this as the *(categorical) Cauchy product*[§]. However, we first require the following preliminary definition:

Definition 4.1. *locally countable categories*
We say that a category $\mathcal{D}$ is **locally countable** when, for all $U, V \in Ob(\mathcal{D})$, the homset $\mathcal{D}(U,V)$ is a countable set. We denote the full subcategory of **Cat**, whose objects are locally countable categories, by **cCat**.

Definition 4.2. Given a PCM-category $\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$ and a locally countable category $\mathcal{D} \in Ob(\mathbf{cCat})$, we define their **Cauchy product** $\mathcal{C}[\mathcal{D}] \in Ob(\mathbf{Cat}_\Sigma)$ as follows:

— **Objects** $Ob(\mathcal{C}[\mathcal{D}]) = Ob(\mathcal{C}) \times Ob(\mathcal{D})$
— **Arrows** The homset $\mathcal{C}[\mathcal{D}]((X,U),(Y,V))$ consists of all functions

$$f : \mathcal{D}(U,V) \to \mathcal{C}(X,Y)$$

such that $\{f(a) \in \mathcal{C}(X,Y)\}_{a\in\mathcal{D}(U,V)}$ is a summable family.
— **Composition** Given $g \in \mathcal{C}[\mathcal{D}]((Y,V),(Z,W))$ and $f \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))$ as functions

$$f : \mathcal{D}(U,V) \to \mathcal{C}(X,Y) \quad \text{and} \quad g : \mathcal{D}(V,W) \to \mathcal{C}(Y,Z)$$

§ Some new terminology is certainly needed. Starting from the theory of monoid semirings, we will replace both monoids and semirings with categories. However, we wish to avoid replacing the term 'monoid-semiring' by 'category-category'.

then $gf \in \mathcal{C}[\mathcal{D}]((X,U),(Z,W))$ is the function from $\mathcal{D}(U,W)$ to $\mathcal{C}(X,Z)$ given by:

$$gf(c) = \sum_{\{(b,a):c=ba\}\subseteq \mathcal{D}(V,W)\times\mathcal{D}(U,V)} g(b)f(a)$$

For clarity, we will often use the shorthand notation

$$gf(c) = \sum_{c=ba} g(b)f(a)$$

— **Summation** An indexed family $\{f_i \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))\}_{i\in I}$ is summable exactly when

$$\{f_i(h) \in \mathcal{C}(X,Y)\}_{(i,h)\in I\times\mathcal{D}(U,V)} \quad \text{is summable in } \mathcal{C}$$

in which case

$$\left(\sum_{i\in I} f_i\right)(h) \overset{def.}{=} \sum_{i\in I} f_i(h) \ \in \ \mathcal{C}(X,Y)$$

Terminology In the above definition of the Cauchy product $\mathcal{C}[\mathcal{D}]$, we refer to the PCM-category $\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$ as the **base category** and the locally countable category $\mathcal{D} \in Ob(\mathbf{cCat})$ as the **index category**.

We now prove that the above construction is well-defined:

Theorem 4.3. The Cauchy product $\mathcal{C}[\mathcal{D}]$ defined above is a **PCM**-category.

Proof. We first show that $\mathcal{C}[\mathcal{D}]$ *is a category, and then consider the indexed summation on homsets.*
We demonstrate that the composition of $\mathcal{C}[\mathcal{D}]$ is well-defined, associative, and has identities:

1 **Composition is well-defined** Given arrows

$$g \in \mathcal{C}[\mathcal{D}]((Y,V),(Z,W)) \quad \text{and} \quad f \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))$$

(i.e. functions $f : \mathcal{D}(U,V) \to \mathcal{C}(X,Y)$ and $g : \mathcal{D}(V,W) \to \mathcal{C}(Y,Z)$ where $\{f(a) \in \mathcal{C}(X,Y)\}_{a\in\mathcal{D}(U,V)}$ and $\{g(b) \in \mathcal{C}(Y,Z)\}_{b\in\mathcal{D}(V,W)}$ are summable), we need to show that $(gf)(c) \in \mathcal{C}(X,Z)$ exists, for all $c \in \mathcal{D}(U,W)$, and

$$\left\{ gf(c) = \sum_{c=ba} g(b)f(a) \right\}_{c\in\mathcal{C}(X,Z)}$$

is also a summable family. By definition, $\sum_{a\in\mathcal{D}(U,V)} f(a) \in \mathcal{C}(X,Y)$ exists, as does $\sum_{b\in\mathcal{D}(V,W)} g(b) \in \mathcal{C}(Y,Z)$. The strong distributivity property thus implies the summability of the indexed family

$$P = \{g(b)f(a)\}_{(b,a)\in\mathcal{D}(V,W)\times\mathcal{D}(U,V)}$$

together with the identity

$$\left(\sum_{b\in\mathcal{D}(V,W)} g(b)\right)\left(\sum_{a\in\mathcal{D}(U,V)} f(a)\right) = \sum (P)$$

Given some arbitrary $c \in \mathcal{D}(U,W)$, consider the (possibly empty) subfamily of P_c of P given by $\{g(b)f(a)\}_{ba=c}$. This is a subfamily of P, and thus is itself a summable family, by the subfamilies property of Proposition 2.4. Therefore $(gf)(c) \in \mathcal{C}(X,Z)$ is well-defined, for all $c \in \mathcal{D}(U,W)$.

Finally, consider the family $\{P_c\}_{c\in\mathcal{D}(U,W)}$. Observe that, for distinct $x \neq y \in \mathcal{D}(U,W)$, the intersection of P_x and P_y is empty. Thus, $\{P_c\}_{c\in\mathcal{D}(U,W)}$ is a partition of the summable family P, and by the weak partition-associativity axiom is itself a summable family satisfying

$$\sum_{c\in\mathcal{D}(U,W)} P_c = \sum_{(b,a)\in\mathcal{D}(V,W)\times\mathcal{D}(U,V)} g(b)f(a)$$

2 **Associativity of composition**

Consider arrows

— $h \in \mathcal{C}[\mathcal{D}]((Z,W),(T,P))$,

— $g \in \mathcal{C}[\mathcal{D}]((Y,V),(Z,W))$,

— $f \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))$.

By definition,

$$(hg)(r) = \sum_{\{(q,p):r=qp\}\subseteq\mathcal{D}(W,P)\times\mathcal{D}(V,W)} h(q)g(p)$$

and similarly,

$$(gf)(c) = \sum_{\{(b,a):c=ba\}\subseteq\mathcal{D}(V,W)\times\mathcal{D}(U,V)} g(b)f(a)$$

Therefore, for all $\gamma \in \mathcal{D}(U,P)$,

$$(h(gf))\,(\gamma) = \sum_{\{(\beta,\alpha):\gamma=\beta\alpha\}\subseteq\mathcal{D}(W,P)\times\mathcal{D}(U,W)} h(\beta)\,(gf)\,(\alpha)$$

which, by definition of the composite $gf \in \mathcal{C}[\mathcal{D}]((X,U),(Z,W))$ is given by

$$(h(gf))\,(\gamma) = \sum_{\{(\beta,\alpha):\gamma=\beta\alpha\}\subseteq\mathcal{D}(W,P)\times\mathcal{D}(U,W)} h(\beta)\left(\sum_{\{(b,a):\alpha=ba\}\subseteq\mathcal{D}(V,W)\times\mathcal{D}(U,V)} g(b)f(a)\right)$$

By distributivity (Proposition 3.2) we may write this as

$$(h(gf))\,(\gamma) = \sum_{\{(\beta,\alpha):\gamma=\beta\alpha\}\subseteq\mathcal{D}(W,P)\times\mathcal{D}(U,W)} \left(\sum_{\{(b,a):\alpha=ba\}\subseteq\mathcal{D}(V,W)\times\mathcal{D}(U,V)} h(\beta)g(b)f(a)\right)$$

Conversely, $((hg)f) \in \mathcal{C}[\mathcal{D}]((X,U),(T,P))$ is given by, for all $\nu \in \mathcal{D}(U,P)$,

$$((hg)f)(\nu) = \sum_{\{(\mu,\lambda):\nu=\mu\lambda\} \subseteq \mathcal{D}(V,P)\times\mathcal{D}(U,V)} (hg)(\mu)f(\lambda)$$

which, by definition of the composite $hg \in \mathcal{C}[\mathcal{D}]((Y,V),(T,P))$ is given by

$$((hg)f)(\nu) = \sum_{\{(\mu,\lambda):\nu=\mu\lambda\} \subseteq \mathcal{D}(V,P)\times\mathcal{D}(U,V)} \left(\sum_{\{(c,b):\mu=cb\} \subseteq \mathcal{D}(W,P)\times\mathcal{D}(V,W)} h(c)g(b) \right) f(\lambda)$$

Again by distributivity (Proposition 3.2) this may be written as

$$((hg)f)(\nu) = \sum_{\{(\mu,\lambda):\nu=\mu\lambda\} \subseteq \mathcal{D}(V,P)\times\mathcal{D}(U,V)} \left(\sum_{\{(c,b):\mu=cb\} \subseteq \mathcal{D}(W,P)\times\mathcal{D}(V,W)} h(c)g(b)f(\lambda) \right)$$

Now observe that, by definition of arrows of $\mathcal{C}[\mathcal{D}]$, the families

— $\{h(c) \in \mathcal{C}(Z,T)\}_{c\in\mathcal{D}(W,P)}$

— $\{g(b) \in \mathcal{C}(Y,Z)\}_{b\in\mathcal{D}(V,W)}$

— $\{f(a) \in \mathcal{C}((X,Y)\}_{a\in\mathcal{D}(U,V)}$

are all summable. Therefore, by the strong distributivity property, the family

$$\{h(c)g(b)f(a)\}_{(c,b,a)\in\mathcal{D}(W,P)\times\mathcal{D}(V,W)\times\mathcal{D}(U,V)}$$

is summable. Given arbitrary $d \in \mathcal{D}(U,P)$, let Q_d be the subfamily of the above indexing set given by

$$Q_d = \{(c,b,a) : d = cba\} \subseteq \mathcal{D}(W,P)\times\mathcal{D}(V,W)\times\mathcal{D}(U,V)$$

Then by the summable subfamilies property, this is summable. By the weak partition-associativity axiom, we may partition $\sum_{Q_d} h(c)g(b)f(a)$ in two distinct ways — by relabelling indices these may be seen to correspond to $((hg)f)(d)$ and $(h(gf))(d)$ respectively. Thus $((hg)f)(d) = (h(gf))(d)$, for all $d \in \mathcal{D}(U,P)$, and thus $(hg)f = h(gf) \in \mathcal{C}[\mathcal{D}]((X,U),(T,P))$, as required.

3 **Identity arrows** Recall the existence of zero elements in a PCM, from Proposition 2.4, and the proof that PCM-categories have zero arrows, in Corollary 3.3. At an object $(X,U) \in Ob(\mathcal{C}[\mathcal{D}])$, the identity arrow is given by $1_{(X,U)}$ by

$$1_{(X,U)}(r) = \begin{cases} 1_X \in \mathcal{C}(X,X) & r = 1_U \in \mathcal{D}(U,U) \\ 0_X & \text{otherwise.} \end{cases}$$

From the definition of composition, and Proposition 2.4, for all $g \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))$ and $f \in \mathcal{C}[\mathcal{D}]((W,T),(X,U))$,

$$\left(g1_{(X,U)}\right)(s) = g(s) \quad \forall s \in \mathcal{D}(U,V)$$

and

$$\left(1_{(X,U)}f\right)(r) = f(r) \quad \forall r \in \mathcal{D}(T,U)$$

Thus $1_{(X,U)} \in \mathcal{C}[\mathcal{D}]((X,U),(X,U))$ is the identity, as required.

It now remains to show that $\mathcal{C}[\mathcal{D}]$ is not only a category, but a PCM-category:

1 **Hom-sets are PCMs** Given objects $(X,U),(Y,V) \in Ob(\mathcal{C}[\mathcal{D}])$, we now demonstrate the summation given in Definition 4.2 above gives a PCM structure to $\mathcal{C}[D]((X,U),(Y,V))$.

— **The unary sum axiom**
Consider an indexed family

$$\{f_i \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))\}_{i\in\{i'\}} \quad \text{where} \quad f_{i'} = f.$$

We first demonstrate that $\{f_i(a) \in \mathcal{C}(X,Y)\}_{(i,a)\in\{i'\}\times\mathcal{D}(U,V)}$ is summable in $\mathcal{C}$. As $\{i'\}$ is a single-element set, $\{i'\}\times\mathcal{D}(U,V) \cong \mathcal{D}$, and (trivially) $f_i = f$, for all $i \in \{i'\}$. Therefore, by Proposition 2.3 the summability of $\{f_i(a) \in \mathcal{C}(X,Y)\}_{(i,a)\in\{i'\}\times\mathcal{D}(U,V)}$ is equivalent to the summability of $\{f(a) \in \mathcal{C}(X,Y)\}_{a\in\mathcal{D}(U,V)}$, and this is summable by the definition of arrows in $\mathcal{C}[\mathcal{D}]$. Thus singleton families are summable in $\mathcal{C}[\mathcal{D}]((X,U),(Y,V))$. Finally, by definition of the summation of $\mathcal{C}[\mathcal{D}]$, and the unary sum axiom for the PCM $\left(\mathcal{C}(X,Y),\Sigma^{X,Y}\right)$,

$$\left(\sum_{i\in\{i'\}} f_i\right)(a) = \sum_{i\in\{i'\}} f_i(a) = f(a)$$

and therefore $\sum_{i\in\{i'\}} f_i = f \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))$.

— **Weak partition-associativity**
Consider a summable family $\{f_i \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))\}_{i\in I}$, and let $\{I_j\}_{j\in J}$ be a partition of I. By definition of summability in $\mathcal{C}[\mathcal{D}]$, the family $\{f_i(a) \in \mathcal{C}(X,Y)\}_{(i,a)\in I\times\mathcal{D}(U,V)}$ is summable in $\mathcal{C}$. Now consider the family $\{f_{i'}(a) \in \mathcal{C}(X,Y)\}_{(i',a)\in I_j\times\mathcal{D}(U,V)}$. This is a subfamily of a summable family of $\mathcal{C}(X,Y)$ and thus, by the summable subfamilies property (Proposition 2.4) is itself a summable family. Therefore, by definition of summability in $\mathcal{C}[\mathcal{D}]$, the family $\{f_{i'} \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))\}_{i\in I'}$ is summable.
Similarly, to show that $\sum_{j\in J}\left(\sum_{i'\in I_j} f_{i'}\right)$ is summable in $\mathcal{C}[\mathcal{D}]((X,U),(Y,V))$, note that $\{\sum_{i'\in I_j} f_{i'}(a) \in \mathcal{C}(X,Y)\}_{(j,a)\in J\times\mathcal{D}(U,V)}$ is summable, by the weak partition-associativity axiom for the PCM $\left(\mathcal{C}(X,Y),\Sigma^{X,Y}\right)$, and (again, by WPA),

$$\left(\sum_{j\in J}\left(\sum_{i'\in I_J} f_{i'}\right)\right)(a) = \left(\sum_{i\in I} f_i\right)(a) \ \in \mathcal{C}(X,Y)$$

for all $a \in \mathcal{D}(U,V)$, and thus

$$\left(\sum_{j\in J}\left(\sum_{i'\in I_J} f_{i'}\right)\right) = \left(\sum_{i\in I} f_i\right) \ \in \ \mathcal{C}[\mathcal{D}]((X,U),(Y,V))$$

Therefore, the summation on $\mathcal{C}[\mathcal{D}]((X,U),(Y,V))$ satisfies weak partition-associativity.

2 **The strong distributive law**
Consider summable families of $\mathcal{C}[\mathcal{D}]$

$$\{f_i \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))\}_{i\in I} \quad \text{and} \quad \{g_j \in \mathcal{C}[\mathcal{D}]((Y,V),(Z,W))\}_{j\in J}$$

Summability of these families is equivalent to the summability of the following families in $\mathcal{C}$

$$\{f_i(a) \in \mathcal{C}(X,Y)\}_{(i,a)\in I\times\mathcal{D}(U,V)} \quad \text{and} \quad \{g_j(b) \in \mathcal{C}(Y,Z)\}_{(j,b)\in J\times\mathcal{D}(V,W)}$$

By the strong distributivity law for $\mathcal{C}$, the following family is therefore summable:

$$\{g_j(b)f_i(a) \in \mathcal{C}(X,Z)\}_{(j,b,i,a)\in J\times\mathcal{D}(V,W)\times I\times\mathcal{D}(U,V)}$$

For all $c \in \mathcal{D}(U,W)$, consider the (possibly empty) subset

$$P_c = \{(j,b,i,a) : c = ba\} \subseteq J \times \mathcal{D}(V,W) \times I \times \mathcal{D}(U,V)$$

Note that $P_c \cap P_{c'} = \emptyset$, for all $c \neq c'$, and

$$\bigcup_{c\in\mathcal{D}(U,W)} P_c = J \times \mathcal{D}(V,W) \times I \times \mathcal{D}(U,V)$$

giving a $\mathcal{D}(U,W)$-indexed partition of the summable family

$$\{g_j(b)f_i(a) \in \mathcal{C}(X,Z)\}_{(j,b,i,a)\in J\times\mathcal{D}(V,W)\times I\times\mathcal{D}(U,V)}$$

Thus, by the weak partition-associativity property of $\left(\mathcal{C}(X,Z),\Sigma^{X,Y}\right)$, the family

$$\{g_j(b)f_i(a) \in \mathcal{C}(X,Z) : ba = c\}_{(j,c,i)\in J\times\mathcal{D}(U,W)\times I}$$

is summable, demonstrating that $\{g_jf_i \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))\}_{(j,i)\in J\times I}$ is summable in $\mathcal{C}[\mathcal{D}]$, as required.
For all $c \in \mathcal{D}(U,W)$, the identity

$$\left(\sum_{j\in J} g_j\right)\left(\sum_{i\in I} f_i\right)(c) = \left(\sum_{(j,i)\in J\times I} g_jf_i\right)(c) \ \in \mathcal{C}(X,Z)$$

is then immediate from the existence of both sides of this equation, and the strong distributivity law for $\mathcal{C}$, and so

$$\left(\sum_{j\in J} g_j\right)\left(\sum_{i\in I} f_i\right) = \left(\sum_{(j,i)\in J\times I} g_jf_i\right) \ \in \mathcal{C}[\mathcal{D}]((X,U),(Z,W))$$

as required.

□

4.1. *The Cauchy product as a bifunctor*

Theorem 4.4. The Cauchy product of Definition 4.2 defines a bifunctor

$$(\_)[\_] : \mathbf{Cat}_\Sigma \times \mathbf{cCat} \to \mathbf{Cat}_\Sigma$$

That is

1 Given $\mathcal{D} \in Ob(\mathbf{cCat})$, then $(\_)[\mathcal{D}] : \mathbf{Cat}_\Sigma \to \mathbf{Cat}_\Sigma$ is a functor.
2 Given $\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$, then $\mathcal{C}[\_] : \mathbf{cCat} \to \mathbf{Cat}_\Sigma$ is a functor.

Proof.

1 We first demonstrate that for arbitrary $\mathcal{D} \in \mathbf{cCat}$, the map $(\_)[\mathcal{D}] : \mathbf{Cat}_\Sigma \to \mathbf{Cat}_\Sigma$ defines a functor.

— **on Objects** Given a PCM-category $\mathcal{C} \in Ob(\mathcal{C})$, then $\mathcal{C}[\mathcal{D}] \in Ob(\mathbf{Cat}_\Sigma)$ is as defined in Definition 4.2.

— **on Arrows** Given $\Gamma \in \mathbf{Cat}_\Sigma(\mathcal{C},\mathcal{E})$, we define the functor $(\Gamma[\mathcal{D}]) \in \mathbf{Cat}_\Sigma(\mathcal{C}[\mathcal{D}],\mathcal{E}[\mathcal{D}])$ as follows:

- *on Objects* For all $(X,U) \in Ob(\mathcal{C}[\mathcal{D}])$, we define $(\Gamma[\mathcal{D}])\,(X,U) = (\Gamma(X),U)$.
- *on Arrows* Given an arbitrary arrow $f \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))$, we define $(\Gamma[\mathcal{D}])\,(f) \in \mathcal{E}[\mathcal{D}]((\Gamma(X),U),(\Gamma(Y),V))$ by, for all $r \in \mathcal{D}(U,V)$,

$$(\Gamma[\mathcal{D}])\,(f)(r) = \Gamma(f(r)) \in \mathcal{E}(\Gamma(X),\Gamma(Y))$$

It is immediate that this is well-defined as an arrow in $\mathcal{E}[\mathcal{D}]((\Gamma(X),U),(\Gamma(Y),V))$ since, as Γ is a **PCM**-functor (i.e. an arrow of $\mathbf{Cat}_\Sigma(\mathcal{C},\mathcal{E})$)

$$\sum_{r\in\mathcal{D}(U,V)} f(r) \ \text{ exists } \quad \Rightarrow \quad \sum_{r\in\mathcal{D}(U,V)} \Gamma(f(r)) \ \text{ exists.}$$

To prove compositionality, consider

$$f \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V)) \ \text{ and } \ g \in \mathcal{C}[\mathcal{D}]((Y,V),(Z,W))$$

By definition of composition, $gf(c) = \sum_{c=ba} g(b)f(a)$, for all $c \in \mathcal{D}(U,W)$. However, by definition of the functor $(\Gamma[\mathcal{D}])$,

$$((\Gamma[\mathcal{D}])\,(g)\,(\Gamma[\mathcal{D}])\,(f))\,(c) = \sum_{c=ba} \Gamma(g(b))\Gamma(f(a))$$

By functoriality of Γ,

$$((\Gamma[\mathcal{D}])\,(g)\,(\Gamma[\mathcal{D}])\,(f))\,(c) = \sum_{c=ba} \Gamma(g(b)f(a))$$

and as Γ is a PCM-functor,

$$((\Gamma[\mathcal{D}])\,(g)\,(\Gamma[\mathcal{D}])\,(f))\,(c) = \Gamma\left(\sum_{c=ba} g(b)f(a)\right) = (\Gamma[\mathcal{D}])\,(gf)(c)$$

Finally, given another functor $\Delta \in \mathbf{Cat}_\Sigma(\mathcal{E},\mathcal{F})$, then

- On objects:

$$(\Delta[\mathcal{D}])\,(\Gamma[\mathcal{D}])\,(X,U) = (\Delta\Gamma(X),U) = ((\Delta\Gamma)[\mathcal{D}])\,(X,U)$$

- On arrows: given $f \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))$, then

$$(\Delta[\mathcal{D}])\,(\Gamma[\mathcal{D}])\,(f)(r) = \Delta(\Gamma(f))(r) = (\Delta\Gamma(f))(r) = ((\Delta\Gamma)[\mathcal{D}])\,(f)(r)$$

2 We now demonstrate that for arbitrary $\mathcal{C} \in \mathbf{Cat}_\Sigma$, the map $\mathcal{C}[\_] : \mathbf{cCat} \to \mathbf{Cat}_\Sigma$ is also functorial.

— **on Objects** Given arbitrary $\mathcal{D} \in Ob(\mathbf{cCat})$, then $\mathcal{C}[\mathcal{D}]$ is given in Definition 4.2.

— **on Arrows** Given a functor $\Lambda \in \mathbf{cCat}(\mathcal{D}, \mathcal{H})$, we define $\mathcal{C}[\Lambda] \in \mathbf{Cat}_\Sigma(\mathcal{C}[\mathcal{D}], \mathcal{C}[\mathcal{H}])$ by:

- *on Objects* $\mathcal{C}[\Lambda](X, U) = (X, \Lambda(U))$.
- *on Arrows*, given $f \in \mathcal{C}[\mathcal{D}]((X, U), (Y, V))$, we define

$$(\mathcal{C}[\Lambda])(f) \in \mathcal{C}[\mathcal{H}]((X, \Lambda(U)), (Y, \Lambda(V)))$$

by, for all $x \in \mathcal{H}(\Lambda(U), \Lambda(V))$,

$$(\mathcal{C}[\Lambda])(f)(x) = \sum_{x=\Lambda(a)} f(a) \ \in \mathcal{C}(X, Y)$$

The above sum is well-defined, since $\{f(a)\}_{a \in \mathcal{D}(U,V)}$ is a summable family. Also, note that $\{b = \Gamma(a)\}_{x \in \mathcal{H}(\Gamma(U), \Gamma(V))}$ is a partition of $\mathcal{D}(U, V)$, and thus, by the weak partition-associativity axiom, $\{(\mathcal{C}[\Lambda])(f)(x)\}_{x \in \mathcal{H}(\Lambda(U), \Lambda(V))}$ is summable, and so $(\mathcal{C}[\Lambda])(f) \in \mathcal{C}[\mathcal{H}]((X, \Lambda(U)), (Y, \Lambda(V)))$ is well-defined.

To prove compositionality, consider

$$f \in \mathcal{C}[\mathcal{D}]((X, U), (Y, V)) \ \text{ and } \ g \in \mathcal{C}[\mathcal{D}]((Y, V), (Z, W))$$

By definition of composition, $gf(c) = \sum_{c=ba} g(b)f(a)$, for all $c \in \mathcal{D}(U, W)$, and hence, for all $z \in \mathcal{H}(\Lambda(U), \Lambda(W))$,

$$(\mathcal{C}(\Lambda)(gf))(z) = \sum_{z=\Lambda(c)} (gf)(c)$$

Now note that, for all $y \in \mathcal{H}(\Lambda(V), \Lambda(W))$ and $x \in \mathcal{H}(\Lambda(U), \Lambda(V))$,

$$(\mathcal{C}(\Lambda)(g))(y) = \sum_{y=\Lambda(b)} g(b) \quad \text{and} \quad (\mathcal{C}(\Lambda)(f))(x) = \sum_{x=\Lambda(a)} f(c)$$

and thus, by the strong distributive law for PCM-categories, and the functoriality of Λ,

$$(\mathcal{C}(\Lambda)(g))(\mathcal{C}(\Lambda)(f))(z) = \sum_{z=\Lambda(c)} (gf)(c) = (\mathcal{C}(\Lambda)(gf))(z) \ \in \mathcal{C}(X, Z)$$

Finally, given another functor $\Omega \in \mathbf{cCat}(\mathcal{H}, \mathcal{K})$, then

— On objects:

$$(\mathcal{C}[\Omega])(\mathcal{C}[\Lambda])(X, U) = (X, \Omega\Lambda(U)) = (\mathcal{C}[\Omega\Lambda])(X, U)$$

— On arrows: given $f \in \mathcal{C}[\mathcal{D}]((X, U), (Y, V))$, then for all $p \in \mathcal{K}(\Omega\Lambda(U), \Omega\Lambda(V))$,

$$(\mathcal{C}[\Omega])(\mathcal{C}[\Lambda])(f)(p) = \sum_{p=\Omega(x)\ ,\ x=\Lambda(a)} f(a) = \sum_{p=\Omega\Lambda(a)} f(a) = (\mathcal{C}[\Omega\Lambda])(f)(p)$$

□

Is the Cauchy product a monoidal tensor? Since the Cauchy product is a bifunctor $\mathbf{Cat}_\Sigma \times \mathbf{cCat} \rightarrow \mathbf{Cat}_\Sigma$, it is natural to wonder whether, when restricted to locally countable PCM-categories, it is in fact a monoidal tensor. It is also easy to show that this is not the case: consider three locally countable PCM-categories, $\mathcal{C}, \mathcal{D}, \mathcal{E} \in Ob(\mathbf{Cat}_\Sigma)$, and denote their (object-indexed families of) summations by $\Sigma^{\mathcal{C}(\_,\_)}$, $\Sigma^{\mathcal{D}(\_,\_)}$, $\Sigma^{\mathcal{E}(\_,\_)}$ respectively. Then it is immediate that the structure of $\mathcal{C}[\mathcal{D}[\mathcal{E}]]$ depends on the family of summations $\Sigma^{\mathcal{D}(\_,\_)}$ on the homsets of $\mathcal{D}$, whereas the structure of $(\mathcal{C}[\mathcal{D}])[\mathcal{E}]$ is independent of $\Sigma^{\mathcal{D}(\_,\_)}$. Therefore, in general, $(\mathcal{C}[\mathcal{D}])[\mathcal{E}]$ cannot be equal to $\mathcal{C}[\mathcal{D}[\mathcal{E}]]$, even up to a canonical isomorphism.

Rather, as we now demonstrate, there exist embeddings of $\mathcal{C}$ into $\mathcal{C}[\mathcal{D}]$ indexed by objects of $\mathcal{D}$, together with embeddings of $\mathcal{D}$ into $\mathcal{C}[\mathcal{D}]$ indexed by objects of $\mathcal{C}$. The embeddings of $\mathcal{C}$ into $\mathcal{C}[\mathcal{D}]$ also have a common left-inverse, giving an indexed family of retractions.

5. Embedding the base category into a Cauchy product

We now give an embedding of the base category $\mathcal{C}$ into the Cauchy product $\mathcal{C}[\mathcal{D}]$, and show that $\mathcal{C}$ is a retract of $\mathcal{C}[\mathcal{D}]$.

We first exhibit a forgetful functor from $\mathcal{C}[\mathcal{D}]$ to $\mathcal{C}$:

Definition 5.1. Given $\mathcal{D} \in Ob(\mathbf{cCat})$, and $\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$, we define $\sigma_{\mathcal{C},\mathcal{D}} : \mathcal{C}[\mathcal{D}] \rightarrow \mathcal{C}$ by

— **(on objects)** $\sigma_{\mathcal{C},\mathcal{D}}(X, U) = X$, for all $(X, U) \in Ob(\mathcal{C}[\mathcal{D}])$.
— **(on arrows)** Given $h \in \mathcal{C}[D]((X, U), (Y, V))$, then

$$\sigma_{\mathcal{C},\mathcal{D}}(h) = \sum_{a \in \mathcal{D}(U,V)} h(a) \ \in \ \mathcal{C}(X, Y)$$

Proposition 5.2. $\sigma_{\mathcal{C},\mathcal{D}} : \mathcal{C}[\mathcal{D}] \rightarrow \mathcal{C}$, as given above, is a PCM-functor.

Proof. First note that, by definition of arrows in $\mathcal{C}[\mathcal{D}]$, the family $\{h(a)\}_{a \in \mathcal{D}(U,V)}$ is summable for all $h \in \mathcal{C}[D]((X, U), (Y, V))$, and so $\sigma_{\mathcal{C},\mathcal{D}}(h) = \sum_{a \in \mathcal{D}(U,V)} h(a) \ \in \ \mathcal{C}(X, Y)$ is well-defined.

To prove functoriality, consider $k \in \mathcal{C}[\mathcal{D}]((Y, V), (Z, W))$. Then

$$\sigma(k)\sigma(h) = \left(\sum_{b \in \mathcal{D}(V,W)} k(b) \right) \left(\sum_{a \in \mathcal{D}(U,V)} h(a) \right)$$

By the strong distributivity property,

$$\sigma(k)\sigma(h) = \sum_{(b,a) \in \mathcal{D}(V,W) \times \mathcal{D}(U,V)} k(b)h(a)$$

Conversely, $kh \in \mathcal{C}[\mathcal{D}]((X, U)(Z, W))$ is defined by, for all $c \in \mathcal{D}(U, W)$,

$$(kh)(c) \sum_{c=ba} k(b)h(a)$$

Now note that $\{(kh)(c)\}_{c\in\mathcal{D}(U,W)}$ is a summable family, and by weak partition-associativity,

$$\sigma(kh) = \sum_{c=ba} k(b)h(a) = \sum_{(b,a)\in\mathcal{D}(V,W)\times\mathcal{D}(U,V)} k(b)h(a) = \sigma(k)\sigma(h)$$

Thus $\sigma : \mathcal{C}[\mathcal{D}] \to \mathcal{C}$ preserves composition. The proof that it also preserves identities follows from the formula for identities in PCM-categories given in Theorem 4.3,

$$1_{(X,U)}(r) = \begin{cases} 1_X \in \mathcal{C}(X,X) & r = 1_U \in \mathcal{D}(U,U) \\ 0_X & \text{otherwise.} \end{cases}$$

It is immediate that $\sigma\left(1_{(X,U)}\right) = 1_X \in \mathcal{C}(X,X)$.

Now consider a summable family $\{f_i \in \mathcal{C}[\mathcal{D}]((X,U),(Y,V))\}_{i\in I}$. By definition of summation in $\mathcal{C}[\mathcal{D}]$, the family $\{f_i(a) \in \mathcal{C}(X,Y)\}_{(i,a)\in I\times\mathcal{D}(U,V)}$ is summable in $\mathcal{C}$, and again by definition

$$\left(\sum_{i\in I} f_i\right)(a) \;=\; \sum_{i\in I} f_i(a) \;\in\; \mathcal{C}(X,Y)$$

Thus, by weak partition-associativity, and Proposition 2.3,

$$\sum_{i\in I}\sigma(f_i) = \sum_{i\in I}\left(\sum_{a\in\mathcal{D}(U,V)} f(a)\right) = \sum_{a\in\mathcal{D}(U,V)}\left(\sum_{i\in I} f_i(a)\right) = \sigma\left(\sum_{i\in I} f_i\right)$$

Therefore, $\sigma : \mathcal{C}[\mathcal{D}] \to \mathcal{C}$ is a PCM-functor. □

We now exhibit a family of embeddings of $\mathcal{C}$ into $\mathcal{C}[\mathcal{D}]$, indexed by objects of $\mathcal{D}$:

Definition 5.3. Let $\mathcal{D}$ be an arbitrary category, and let $\mathcal{C}$ be a PCM-category. For all $U \in Ob(\mathcal{D})$, we define $\eta_{\mathcal{C},U} : \mathcal{C} \to \mathcal{C}[\mathcal{D}]$ by

— **(on objects)** $\eta_{\mathcal{C},U}(X) = (X,U)$, for all $X \in Ob(\mathcal{C})$.
— **(on arrows)** Given $h \in \mathcal{C}(X,Y)$, then $\eta_{\mathcal{C},U}(h) \in \mathcal{C}[\mathcal{D}]((X,U),(Y,U))$ is the function $\eta_{\mathcal{C},U}(h) : \mathcal{D}(U,U) \to \mathcal{C}(X,Y)$ given by

$$\left(\eta_{\mathcal{C},U}(h)\right)(a) = \begin{cases} h & a = 1_U \\ 0_{XY} & a \neq 1_U \end{cases}$$

We prove that these maps are injective PCM-functors.

Proposition 5.4. For all $U \in Ob(\mathcal{D})$, the map $\eta_{\mathcal{C},U} : \mathcal{C} \to \mathcal{C}[\mathcal{D}]$ defined above is an injective PCM-functor.

Proof. Given $h \in \mathcal{C}(X,Y)$ then $\eta_U(h)$ is trivially well-defined as an arrow of $\mathcal{C}[\mathcal{D}]((X,U),(Y,U))$, since $\sum_{a\in\mathcal{D}(U,U)} h(a) = h$, by Proposition 2.4. Now consider $k \in \mathcal{C}(Y,Z)$. By definition of composition in $\mathcal{C}[\mathcal{D}]$,

$$\left(\eta_{\mathcal{C},U}(k)\right)\left(\eta_{\mathcal{C},U}(h)\right)(c) = \sum_{c=ba}\left(\eta_{\mathcal{C},U}(k)\right)(b)\left(\eta_{\mathcal{C},U}(h)\right)(a)$$

However,

$$(\eta_{\mathcal{C},U}(k))\,(b)\,(\eta_{\mathcal{C},U}(h))\,(a) = \begin{cases} kh & b = a = 1_U \\ 0_{XY} & \text{otherwise} \end{cases}$$

and therefore

$$(\eta_{\mathcal{C},U}(k))\,(\eta_{\mathcal{C},U}(h))\,(c) = \begin{cases} kh & c = 1_U \\ 0_{XY} & c \neq 1_U \end{cases}$$

giving $\eta_{\mathcal{C},U}(k)\eta_{CC,U}(h) = \eta_{\mathcal{C},U}(kh)$ as required. It is immediate from the definition that $\eta_{\mathcal{C},U}(1_X) = 1_{(X,U)} \in \mathcal{C}[\mathcal{D}]((X,U),(X,U))$, for all $X \in Ob(\mathcal{C})$.

For the summation, consider a summable family $\{f_i \in \mathcal{C}(X,Y)\}_{i\in I}$. By definition of summability in $\mathcal{C}[\mathcal{D}]$, the family $\{\eta_{\mathcal{C},U}(f_i) \in \mathcal{C}[\mathcal{D}]\}_{i\in I}$ is also summable, and

$$\sum_{i\in I} \eta_{\mathcal{C},U}(f_i) \;=\; \eta_{\mathcal{C},U}\left(\sum_{i\in I} f_i\right)$$

The injectivity of $\eta_{\mathcal{C},U} : \mathcal{C} \to \mathcal{C}[U]$ on objects is immediate. To demonstrate injectivity on arrows, consider $f, f' \in \mathcal{C}(X,Y)$ satisfying $\eta_{\mathcal{C},U}(f) = \eta_{\mathcal{C},U}(f')$. Then, for all $a \in \mathcal{D}(U,U)$,

$$\eta_{\mathcal{C},U}(f)(a) = \eta_{\mathcal{C},U}(f')(a)$$

Taking $a = 1_U$ gives $f = f'$, as required. □

We therefore have a family of injective PCM-functors from $\mathcal{C}$ to $\mathcal{C}[\mathcal{D}]$ indexed by the objects of $\mathcal{D}$.

Proposition 5.5. Let $\mathcal{D}$ be an arbitrary category, and let $\mathcal{C}$ be a PCM-category. Then there exists a family of retractions from $\mathcal{C}[\mathcal{D}]$ to $\mathcal{C}$, indexed by objects of $\mathcal{D}$.

Proof. For arbitrary $U \in Ob(\mathcal{D})$, we demonstrate that $\sigma_{\mathcal{C},\mathcal{D}}\eta_{\mathcal{C},U} = Id_{\mathcal{C}}$:

— On objects:

$$\sigma_{\mathcal{C},\mathcal{D}}\eta_{\mathcal{C},U}(X) = \sigma_{\mathcal{C},\mathcal{D}}(X,U) = X$$

— On arrows: given $f \in \mathcal{C}(X,Y)$, then

$$\sigma_{\mathcal{C},\mathcal{D}}(\eta_{\mathcal{C},U}(f)) = \sum_{a\in\mathcal{D}(U,U)} (\eta_{\mathcal{C},U}(f)) \quad \text{where } \eta_{\mathcal{C},U}(f)(a) = \begin{cases} f & a = 1_U \\ 0_{XY} & \text{otherwise} \end{cases}$$

and so by proposition 2.4, $\sigma_{\mathcal{C},\mathcal{D}}(\eta_{\mathcal{C},U}(f)) = f \in \mathcal{C}(X,Y)$.

Thus $\sigma_{\mathcal{C},\mathcal{D}} : \mathcal{C}[\mathcal{D}] \to \mathcal{C}$ is left-inverse to <u>all</u> $\eta_{\mathcal{C},U} : \mathcal{C} \to \mathcal{C}[\mathcal{D}]$, and so $\mathcal{C}$ is a retract of $\mathcal{C}[\mathcal{D}]$, with retractions indexed by $U \in Ob(\mathcal{D})$. □

Lie Algebras, Lie Groups, and the exponential map The above PCM-functors $\eta_{\mathcal{C},U} : \mathcal{C} \to \mathcal{C}[\mathcal{D}]$ and $\sigma_{\mathcal{C},\mathcal{D}} : \mathcal{C}[\mathcal{D}] \to \mathcal{C}$ are clearly motivated by the construction of Lie algebras from Lie groups, and the corresponding exponential maps. However, the precise connection is subtle, and requires generalising this theory to the case where index categories are *not* required to be locally countable. A detailed discussion is therefore well beyond the scope of this paper.

6. Embedding the index category into a Cauchy product

Similarly to the embeddings of the base category into a Cauchy product indexed by objects of the index category, we now exhibit a family of embeddings of the index category into a Cauchy product, indexed by objects of the base category:

Definition 6.1. Let $\mathcal{D}$ be an arbitrary category, and let $\mathcal{C}$ be a PCM-category. For all $X \in Ob(\mathcal{C})$ we define the functor $\gamma_{X,\mathcal{D}} : \mathcal{D} \to \mathcal{C}[\mathcal{D}]$ by

— **(On objects)** $\gamma_{X,\mathcal{D}}(U) = (X, U)$, for all $U \in Ob(\mathcal{D})$,

— **(On arrows)** Given $h \in \mathcal{D}(U, V)$, then $\gamma_{X,\mathcal{D}}(h) \in X[\mathcal{D}]((X, U), (X, V))$ is defined by

$$\gamma_{X,\mathcal{D}}(h)(a) = \begin{cases} 1_X & a = h \\ 0_X & \text{otherwise.} \end{cases}$$

Proposition 6.2. $\gamma_{X,\mathcal{D}} : \mathcal{D} \to \mathcal{C}[\mathcal{D}]$,as defined above, is an injective functor for all $X \in Ob(\mathcal{C})$.

Proof. By Proposition 2.4, it is immediate that, for all $h \in \mathcal{D}(U, V)$, the family $\{\gamma_{X,\mathcal{D}}(h)(a) \in \mathcal{C}(X, X)\}_{a\in\mathcal{D}(U,V)}$ is summable, and hence $\gamma_{X,\mathcal{D}}(h)$ is an arrow of $\mathcal{C}[\mathcal{D}]((X, U), (X, V))$. To demonstrate functoriality, consider $k \in \mathcal{D}(U, V)$. Then

$$(\gamma_{X,\mathcal{D}}(k)\gamma_{X,\mathcal{D}}(h))(c) = \sum_{c=ba} \gamma_{X,\mathcal{D}}(k)(b)\gamma_{X,\mathcal{D}}(h)(a)$$

However,

$$\gamma_X(k)(b) \begin{cases} 1_X & b = k \\ 0_{XX} & \text{otherwise} \end{cases}$$

and

$$\gamma_X(h)(a) \begin{cases} 1_X & a = h \\ 0_{XX} & \text{otherwise} \end{cases}$$

Therefore,

$$(\gamma_X(k)\gamma_X(h))(c) = \begin{cases} 1_X & c = kh \\ 0_{XX} & \text{otherwise} \end{cases}$$

and thus $\gamma_{X,\mathcal{D}}(k)\gamma(X, \mathcal{D})(h) = \gamma_{X,\mathcal{D}}(kh)$. The proof that $\gamma_{X,\mathcal{D}}$ also preserves identities is trivial.

To demonstrate injectivity, consider $h, h' \in \mathcal{D}(U, V)$. Then

$$\gamma_X(h)(a) \begin{cases} 1_X & a = h \\ 0_{XX} & \text{otherwise} \end{cases}$$

and

$$\gamma_X(h')(a) \begin{cases} 1_X & a = h' \\ 0_{XX} & \text{otherwise} \end{cases}$$

These are identical exactly when $h = h'$. □

We therefore have a family of injective functors from $\mathcal{D}$ to $\mathcal{C}[\mathcal{D}]$ indexed by objects of $\mathcal{C}$.

7. Cauchy products and Fourier transforms

Now we have established the basic theory of the Cauchy product, we turn our attention to the algebraic description of the Discrete Fourier Transform (DFT) given in (Nicholson 1971), and how we may generalise this theory to our categorical setting. A discrete Fourier transform is defined in (Nicholson 1971) to be an isomorphism between a group ring and a direct sum of rings, as follows:

Definition 7.1. *The discrete Fourier transform*
Given a ring $(R, \times, +)$ and a finite group $(G, \cdot)$, a **Discrete Fourier Transform (DFT)** is an isomorphism

$$\mathfrak{F} : \; R[G] \; \cong \; R^{\oplus n}$$

where $R[G]$ is the group ring given in Definition 1.1, and $R^{\oplus n}$ is the n-fold direct sum of R with itself, where $n \in \mathbb{N}$ is a function of the structure of G.

Such a Fourier transform is not, of course, guaranteed to exist. Necessary and sufficient conditions on the ring R, for the case when G is abelian, are given in (Britten, Lemire 1981), and the (significantly less well understood) non-abelian case is studied in (Terras 1999).

In our general categorical setting, we wish to take the above definition, and replace the group ring $R[G]$ by a Cauchy product $\mathcal{C}[\mathcal{D}]$ and the n-fold direct sum by a product category $\prod^n \mathcal{C}$. However, the requirement that a Fourier transform is an *isomorphism* may well be too strong in the categorical setting, so we make the following definition:

Definition 7.2. *Categorical Fourier transforms*
Given $\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$ and $\mathcal{D} \in Ob(\mathbf{cCat})$, we define a **categorical Fourier transform (CFT)** to be an injective PCM-functor

$$\mathfrak{F} : \mathcal{C}[\mathcal{D}] \longrightarrow \prod^n \mathcal{D}$$

where $n > 0$ is some function of the structure of $\mathcal{D}$.

The study and construction of such functors is an ambitious program, and a full analysis would require (for example) a categorical analogue of the theory of group characters (Ledermann 1987). We therefore consider a special case, based on the simplest possible case from the algebraic theory of discrete Fourier transform. This is intended to act as a prototype for a more general theory.

7.1. *Fourier transforming a category over a prime cyclic group*

The simplest possible case of a discrete Fourier transform, as defined in (Nicholson 1971), arises for group rings of the form $R[\mathbb{Z}_p]$, where $p \geq 2$ is prime. We consider a similar situation in the theory of the Cauchy product, with the intention that this simple case will provide a model for the general situation. We first write the group $\mathbb{Z}_p$ as a (one-object) category, and give general conditions related to the existence of a categorical Fourier transform.

Definition 7.3. *Cyclic Categories, Natural representations*
Given a prime $p \geq 2$, we define the **cyclic category** $\mathcal{Z}_p$ to have a single object $*$, and exactly p distinct arrows generated by some $\omega \in \mathcal{Z}_p(*,*)$. From this definition, it is immediate that $\omega^p = 1_*$, and $\mathcal{Z}_p(*,*) \cong \mathbb{Z}_p$. We will therefore use the notation ω^{-k} for the products $\omega^{n-k} = \omega^{2n-k} = \ldots$

Let $\mathcal{C} \in \mathbf{Cat}_\Sigma$ be an arbitrary PCM-category. We define a **natural representation** of $\mathcal{Z}_p$ in $\mathcal{C}$ to be an indexed family of non-trivial functors¶

$$\chi_A : \mathcal{Z}_p \rightarrow \mathcal{C} \quad \text{for all} \quad A \in Ob(\mathcal{C})$$

satisfying:

1 **(Object Identification)** $\chi_A(U) = A \in Ob(\mathcal{C})$, for all $U \in Ob(\mathcal{Z}_p)$.
2 **(Naturality in A)** $\chi_B(v)f = f\chi_A(v)$, for all $f \in \mathcal{C}(A,B)$ and $v \in \mathcal{Z}_p(U,V)$.
3 **(Zero-Summability)** $\sum_{v \in \mathcal{D}(U,V)} \chi_A(v) = 0_{AA}$.

Of course, these axioms may be stated more simply, given that $\mathcal{Z}_p$ is a one-object category, and in certain cases (notably, when $\mathcal{C}$ is simply the complex plane) axiom 3 (Zero-Summability) follows from the requirement that $\chi\_$ is non-trivial. However, as stated above, we intend this special case to be a model for the general theory. Note that in this special case, the non-triviality requirement, together with the primality of p, implies that each χ_A is an embedding.

Note also that the naturality condition can readily be satisfied in a category with biproducts. Let $\mathcal{B}$ be a category with biproducts, and denote the unit object by I. Then an injective monoid homomorphism $\chi : \mathcal{Z}_p(*,*) \rightarrow \mathcal{B}(I,I)$ suffices. This follows from the theory of *abstract scalars* (Abramsky 2005).

Using the above definitions, we will exhibit a functor from $\mathcal{C}[\mathcal{Z}_p]$ to $\prod^p \mathcal{C}$. This functor will factor through the subcategory of $\prod^p \mathcal{C}$ defined below:

Definition 7.4. *Diagonal product categories*
Given $\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$, and $0 < n \in \mathbb{N}$, we define the **diagonal product** $\Delta^n\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$ to be the full subcategory of the product category $\prod^n \mathcal{C}$ generated by all diagonal objects (i.e. those of the form $A \times A \times \ldots \times A$, for some $A \in Ob(\mathcal{C})$). We use the notation A^n for such objects. All arrows $f \in \Delta^n(\mathcal{C})(X^n, Y^n)$ are of the form $(f_0, f_1, \ldots, f_{n-1})$, where $f_i : X \rightarrow X$. We therefore use the notation $\{f(k)\}_{k=0}^{n-1} : X^n \rightarrow Y^n$ for arrows in this category, with the simple observation that $(gf)(k) = g(k)f(k)$. Finally, the summation in this category is as given in Definition 3.9.

We now exhibit a PCM-functor from $\mathcal{C}[\mathcal{Z}_p]$ to $\Delta^p\mathcal{C}$.

Theorem 7.5. Given $\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$, and a natural representation $\chi\_ : \mathcal{Z}_p \rightarrow \mathcal{C}$, then the map $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \rightarrow \Delta^p\mathcal{C}$, given by

— *(On Objects)* $\mathfrak{F}(A,*) = A^p$

¶ We say a functor $\Gamma : \mathcal{C} \rightarrow \mathcal{D}$ is 'non-trivial' when there exists some $f \neq g \in \mathcal{C}(X,Y)$ such that $\Gamma(f) \neq \Gamma(g)$.

— *(On Arrows)* Given $r \in \mathcal{C}[\mathcal{Z}_p]((A,*),(B,*))$, then $\mathfrak{F}(r) \in \Delta^p\mathcal{C}(A^p,B^p)$ is defined by

$$\mathfrak{F}(r)(a) = \sum_{\alpha\in\mathbb{Z}_p} r(\omega^\alpha)\chi_A(\omega^{-\alpha a}) \quad \text{for all } a \in \{0,...,p-1\}$$

is a PCM-functor.

Proof.
We show that $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \to \Delta^p\mathcal{C}$ is well-defined, functorial, and a PCM-functor:

1 $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \longrightarrow \Delta^p\mathcal{C}$ **is well-defined**
By definition of $\mathcal{Z}_p$, we may see that $\{\omega^\alpha\}_{\alpha\in\mathbb{Z}_p} = \{g\}_{g\in\mathcal{Z}_p(*,*)}$, and thus by definition of the Cauchy product, the family $\{r(\omega^\alpha) \in \mathcal{C}(A,B)\}_{\alpha\in\mathbb{Z}_p}$ is summable. This gives the existence of $\mathfrak{F}(r)(0)$, since

$$\mathfrak{F}(r)(0) = \sum_{\alpha\in\mathbb{Z}_p} r(\omega^\alpha)1_A = \sum_{g\in\mathcal{Z}_p} r(g)$$

When $a \neq 0$, the summability of $\{\chi_A(g)\}_{g\in\mathcal{Z}_p(*,*)}$ (Definition 7.3) implies that $\{\chi_A(\omega^\beta) \in \mathcal{C}(A,A)\}_{\beta\in\{0,\dots,n-1\}}$ is summable. Thus, by the strong distributivity property,

$$\left\{r(\omega^\alpha)\chi_A(\omega^\beta) \in \mathcal{C}(A,B)\right\}_{(\alpha,\beta)\in\mathbb{Z}_p\times\{0,\dots,n-1\}}$$

is summable, and satisfies

$$\left(\sum_{\alpha\in\mathbb{Z}_p} r(\omega^\alpha)\right)\left(\sum_{\beta\in\{0,\dots,p-1\}} \chi_A(\omega^\beta)\right) = \sum_{(\alpha,\beta)\in\mathbb{Z}_p\times\{0,\dots,n-1\}} r(\omega^\alpha)\chi_A(\omega^\beta)$$

Now consider, for arbitrary $a \in \mathbb{Z}_p$, the subfamily of the above index set given by

$$P_a = \{(\alpha,\beta) : \beta = -\alpha a \ (mod\ p)\} \ \subseteq \mathbb{Z}_p \times \{0,...,p-1\}$$

by the summable subfamilies property (Proposition 2.4), $\{r(\omega^\alpha)\chi_A(\omega^\beta)\}_{(\alpha,\beta)\in P_a}$ is summable. However, by definition, $\mathfrak{F}(r)(a) = \sum_{(\alpha,\beta)\in P_a} r(\omega^\alpha)\chi_A(\omega^\beta) \ \in \ \mathcal{C}(A,B)$ and thus $\mathfrak{F}(r) \in \Delta^p(A^p,B^p)$ is well-defined.

2 $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \longrightarrow \Delta^p\mathcal{C}$ **is a functor**
We now show that $\mathfrak{F}$ preserves composition. Consider $s \in \mathcal{C}[\mathcal{Z}_p]((B,*),(C,*))$ and $r \in \mathcal{C}[\mathcal{Z}_p]((A,*),(B,*))$. Then for all $a \in \{0,...,p-1\}$,

$$[\mathfrak{F}(r)](a) = \sum_{\alpha\in\mathbb{Z}_p} r(\omega^\alpha)\chi_A(\omega^{-\alpha a}) \ \ , \ \ [\mathfrak{F}(s)](a) = \sum_{\beta\in\mathbb{Z}_p} s(\omega^\beta)\chi_B(\omega^{-\beta a})$$

By definition of composition in $\Delta^p\mathcal{C}$, and the strong distributivity property for $\mathcal{C}$,

$$[\mathfrak{F}(s)\mathfrak{F}(r)](k) = [\mathfrak{F}(s)](k)[\mathfrak{F}(r)](k) = \sum_{(\beta,\alpha)\in\mathbb{Z}_p\times\mathbb{Z}_p} s(\omega^\beta)\chi_B(\omega^{-k\beta})r(\omega^\alpha)\chi_A(\omega^{-k\alpha})$$

By the naturality condition (Part 2 of Definition 7.3) for $\chi\_$,

$$[\mathfrak{F}(s)\mathfrak{F}(r)](k) = \sum_{(\beta,\alpha)\in\mathbb{Z}_p\times\mathbb{Z}_p} s(\omega^\beta)r(\omega^\alpha)\chi_A(\omega^{-k\beta})\chi_A(\omega^{-k\alpha})$$

and as χ_A is a functor,

$$[\mathfrak{F}(s)\mathfrak{F}(r)](k) = \sum_{(\beta,\alpha)\in\mathbb{Z}_p\times\mathbb{Z}_p} s(\omega^\beta)r(\omega^\alpha)\chi_A(\omega^{-k\beta}\omega^{-k\alpha})$$

By simple modular arithmetic on the exponents of $\omega \in \mathcal{Z}_p(*,*)$,

$$[\mathfrak{F}(s)\mathfrak{F}(r)](k) = \sum_{(\beta,\alpha)\in\mathbb{Z}_p\times\mathbb{Z}_p} s(\omega^\beta)r(\omega^\alpha)\chi_A\left(\omega^{-k(\beta+\alpha)}\right)$$

We now make the substitution $m = \beta+\alpha \ (mod\, p)$ (implicitly appealing to Proposition 2.3), giving

$$[\mathfrak{F}(s)\mathfrak{F}(r)](k) = \sum_{(m,\alpha)\in\mathbb{Z}_p\times\mathbb{Z}_p} s\left(\omega^{m-\alpha\ (mod\ p)}\right) r\left(\omega^\alpha\right)\chi_A\left(\omega^{-km}\right)$$

Now observe that

$$\left\{s\left(\omega^{m-\alpha\ (mod\ p)}\right) r\left(\alpha\right)\right\}_{(m,\alpha)\in\mathbb{Z}_p\times\mathbb{Z}_p} \ = \ \{s\left(\omega^z\right) r\left(\omega^y\right)\}_{m=z+y\ (mod\ p)}$$

Thus, by Proposition 2.3,

$$[\mathfrak{F}(s)\mathfrak{F}(r)](k) = \sum_{m=z+y\ (mod\ p)} s(\omega^z)r(\omega^y)\chi_A\left(\omega^{-km}\right)$$

However, this is $\mathfrak{F}(sr)(k)$, by definition of composition in $\mathcal{C}[\mathcal{Z}_p]$. Thus $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \to \Delta^p\mathcal{C}$ preserves composition.

Finally, recall that the identity of $\mathcal{C}[\mathcal{Z}_p]((A,*),(A,*))$ is the function from $\mathcal{Z}_p(*,*)$ to $\mathcal{C}(A,A)$ given by

$$1_{(A,*)}(f) = \begin{cases} 1_A \in \mathcal{C}(A,A) & f = 1_* \in \mathcal{Z}_p(*,*) \\ 0_A & \text{otherwise.} \end{cases}$$

Then, for all $a \in \{0,...,p-1\}$,

$$\mathfrak{F}\left(1_{(A,*)}\right)(a) = \sum_{\alpha\in\mathbb{Z}_p} 1_{(A,*)}(\omega^\alpha)\chi_A(\omega^{-\alpha a}) \ = \ 1_A \in \mathcal{C}(A,A)$$

3 **$\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \longrightarrow \Delta^p\mathcal{C}$ is a PCM-functor**
We first show that $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \longrightarrow \Delta^p\mathcal{C}$ maps summable families to summable families:

Consider a summable family $\{f_i \in \mathcal{C}[\mathcal{Z}_p]((A,*),(B,*))\}_{i\in I}$. By definition of summability in $\mathcal{C}[\mathcal{Z}_p]$ the family $\{f_i(g) \in \mathcal{C}(A,*)\}_{(i,g)\in I\times\mathcal{Z}_p(*,*)}$ is summable in $\mathcal{C}$.

Now consider the family $\{\mathfrak{F}(f_i) \in \Delta^p(A^p,B^p)\}_{i\in I}$. By definition,

$$\mathfrak{F}(f_i)(a) = \sum_{\alpha\in\mathbb{Z}_p} f_i(\omega^\alpha)\chi_A(\omega^{-\alpha a}) \quad \text{for all } a \in \{0,...,p-1\}$$

and so

$$\{\mathfrak{F}(f_i)(a)\}_{i\in I} = \left\{ \sum_{\alpha\in\mathbb{Z}_p} f_i(\omega^\alpha)\chi_A(\omega^{-\alpha a}) \right\}_{i\in I}$$

We demonstrate that this family is summable: We have already seen that (by definition of summability in $\mathcal{C}[\mathcal{Z}_p]$) the family $\{f_i(g) \in \mathcal{C}(A,B)\}_{(i,g)\in I\times\mathcal{Z}_p}$ is summable. Also, by the summability of $\{\chi_A(g)\}_{g\in\mathcal{Z}_p(*,*)}$ (Definition 7.3), the family $\{\chi_A(\omega^\beta) \in \mathcal{C}(A,A)\}_{\beta\in\{0,\dots,p-1\}}$ is summable. Thus by the strong distributivity property, the family $\{f_i(\omega^\alpha)\chi_A(\omega^\beta) \in \mathcal{C}(A,B)\}_{(i,\alpha,\beta)\in I\times\mathbb{Z}_p\times\{0,\dots,p-1\}}$ is summable.

Consider the subset P_a of the above indexing set given by

$$P_a = \{(i,\alpha,\beta) \ : \ \beta = -\alpha a \ (mod\ p)\} \subseteq I\times\mathbb{Z}_p\times\{0,...,p-1\}$$

By the summable subfamilies property $\{f_i(\omega^\alpha)\chi_A(\omega^\beta)\}_{(i,\alpha,\beta)\in P_a}$ is a summable family. By weak partition-associativity, we may take the sum over α and β, giving that (by definition of $\mathfrak{F}$), the family $\{\mathfrak{F}(f_i)(a) \in \mathcal{C}(A,B)\}_{i\in I}$ is summable. To demonstrate that $\{\mathfrak{F}(f_i) \in \Delta^p\mathcal{C}(A^p,B^p)\}_{i\in I}$ is also summable, note that as $\{\mathfrak{F}(f_i)(a)\}_{i\in I}$ is summable for all $a \in \{0,...,p-1\}$, and the index set (that is, I) is the same for each $a \in \{0,...,n-1\}$, the indexed set $\{\mathfrak{F}(f_i) \in \Delta^p\mathcal{C}(A^p,B^p)\}_{i\in I}$ is summable, by definition of summation in $\Delta^p\mathcal{C}$. Thus $\mathfrak{F}$ preserves summable families.

Finally, to establish equality, recall from Definition 2.1 that weak partition-associativity allows us to equate different partitions of a summable family. We have already established the summability of $\{f_i(\omega^\alpha)\chi_A(\omega^\beta)\}_{(i,\alpha,\beta)\in P_a}$, where

$$P_a = \{(i,\alpha,\beta) \ : \ \beta = -\alpha a \ (mod\ p)\} \subseteq I\times\mathbb{Z}_p\times\{0,...,p-1\}$$

We now use the weak partition-associativity axiom: taking the sum over α and β gives

$$\sum_{P_a} \left(f_i(\omega^\alpha)\chi_A\left(\omega^\beta\right)\right) \ = \ \sum_{i\in I}\mathfrak{F}(f_i)$$

alternatively, taking the sum over I gives

$$\sum_{P_a} \left(f_i(\omega^\alpha)\chi_A(\omega^\beta)\right) \ = \ \mathfrak{F}\left(\sum_{i\in I} f_i\right)$$

giving

$$\left(\sum_{i\in I}\mathfrak{F}(f_i)\right)(a) \ = \ \mathfrak{F}\left(\sum_{i\in I} f(i)\right)(a) \quad \text{for all} \quad a\in\{0,...,p-1\}$$

and thus $\sum_{i\in I}\mathfrak{F}(f_i) \ = \ \mathfrak{F}\left(\sum_{i\in I} f(i)\right)$, and so $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \to \prod^p \mathcal{C}$ is a PCM-functor, as required.

□

Discussion Note that we have not yet proved injectivity, so at this point it is inaccurate to describe $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \to \prod^p \mathcal{C}$ as a categorical Fourier transform. We consider injectivity below; we have paused at this point in order to comment that, so far, we have used neither the full axiom 3 (Zero-Summability) of definition 7.3, nor the primality of p. We have simply required that $\chi_A : \mathcal{Z}_p \to \mathcal{C}$ is natural in $A \in Ob(\mathcal{C})$, and the family $\{\chi_A(g)\}_{g\in\mathcal{Z}(*,*)}$ is summable. Thus a variant of the above theorem is applicable in the restricted forms of PCM-categories commonly used in algebraic program semantics (see Section 10). Despite this, we have exhibited a functor that maps a convolved product to a pointwise product — with all the potential implications this has for complexity of composition (at least, of equivalence classes).

We now unambiguously step outside the usual program semantics framework, and use the additional condition that $\sum_{g\in\mathcal{Z}(*,*)} \chi_A(g) = 0_{AA}$. This allows us to demonstrate a property very similar to the usual orthogonality property in the theory of discrete Fourier transforms. This gives injectivity under very light assumptions, demonstrating that $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \to \prod^p \mathcal{C}$ is indeed a categorical Fourier transform.

Proposition 7.6. The orthogonality property
Let $\mathcal{C}, \Sigma^{(\_,\_)}$ be a PCM-category, and let $\chi_\_ : \mathcal{Z}_p \to \mathcal{C}$ be a natural representation of $\mathcal{Z}_p$, as defined in Definition 7.3. Then, for all $k, l \in \mathbb{N}$, the family $\left\{\chi_A\left(\omega^{(k-l)i}\right)\right\}_{i=0..p-1}$ is summable, and for all $A \in Ob(\mathcal{C})$

$$\sum_{i=0}^{p-1} \chi_A\left(\omega^{(k-l)i}\right) = \begin{cases} \sum_{i=0}^{p-1} 1_A & k = l \\ 0_{AA} & k \neq l \end{cases}$$

Proof.

— **(When $k = l$)** Note that $\omega^{(k-l)i} = \omega^0 = 1_* \in \mathcal{Z}(*,*)$, since arithmetic in the exponents of ω is modulo p. Thus

$$\left\{\chi_A\left(\omega^{(k-l)i}\right)\right\}_{i=0..p-1} = \{\chi_A(1_*)\}_{i=0..p-1} = \{1_A\}_{i=0..p-1}$$

which by Corollary 3.6 is summable and thus

$$\sum_{i=0}^{p-1} \chi_A\left(\omega^{(k-l)i}\right) = \sum_{i=0}^{p-1} 1_A \quad \text{when} \quad k = l$$

— **(When $k \neq l$)** Let us write $\omega^{(k-l)i} = \left(\omega^{(k-l)}\right)^i$. We define $n = k - l \ (mod\ p)$ and note that $\omega^{(k-l)} = \omega^n$. Observe that n and p are trivially co-prime, and hence $\{\omega^i\}_{i=0}^{p-1} = \left\{(\omega^n)^i\right\}_{i=0}^{p-1}$. By definition of a natural representation, $\left\{\chi_A\left(\omega^i\right)\right\}_{i=0..p-1}$ is summable and $\sum_{i=0}^{p-1} \chi_A\left(\omega^i\right) = 0_{AA}$. Therefore,

$$\sum_{i=0}^{p-1} \chi_A\left(\left(\omega^{k-l}\right)^i\right) = \sum_{i=0}^{p-1} \chi_A\left((\omega^n)^i\right) = \sum_{j=0}^{p-1} \chi_A\left(\omega^j\right) = 0_{AA}$$

as required.

□

We now use the above property to exhibit a <u>partial</u> map $\mathfrak{G} : \Delta^p\mathcal{C} \to \mathcal{C}[\mathcal{Z}_p]$. We emphasise that this map need not be defined for all arrows of $\Delta^p\mathcal{C}$ — we will simply show that it is well-defined in the special cases we require (i.e. for arrows of the form $\mathfrak{F}(r)$, for some $r \in \mathcal{C}[\mathcal{Z}_p]((X,*),(Y,*))$).

Definition 7.7. Given an object $X^p \in \Delta^p(\mathcal{C})$, we define $\mathfrak{G}(X^p) = (X,*) \in Ob(\mathcal{C}[\mathcal{Z}_p])$. Given an arrow $u \in \Delta^p\mathcal{C}(A^p, B^p)$, then $\mathfrak{G}(u) \in \mathcal{C}[\mathcal{Z}_p]((A,*),(B,*))$ is defined exactly when the family $\left\{u(\gamma)\chi_A\left(\omega^{k\gamma}\right)\right\}_{(k,\gamma)\in\mathbb{Z}_p\times\{0,..,p-1\}}$ is summable, in which case

$$\mathfrak{G}(u)(\omega^k) = \sum_{\gamma\in\{0,..,p-1\}} u(\gamma)\chi_A\left(\omega^{k\gamma}\right)$$

Theorem 7.8. Given a PCM-category $\mathcal{C} \in Ob(\mathbf{Cat}_\Sigma)$ together with a natural representation $\chi\_ : \mathcal{Z}_p \to \mathcal{C}$, let $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \to \Delta^p\mathcal{C}$ be as defined in Theorem 7.5. Then, for all $r \in \mathcal{C}[\mathcal{Z}_p]((X,*),(Y,*))$,

1 $\mathfrak{G}(\mathfrak{F}(r)) \in \mathcal{C}[\mathcal{Z}_p]$ exists, and
2 $\mathfrak{G}(\mathfrak{F}(r)) = \sum_{i=0}^{p-1} r \in \mathcal{C}[\mathcal{Z}_p]((X,*),(Y,*))$.

Proof. We wish to prove that the sum:

$$\mathfrak{G}(\mathfrak{F}(r))(\omega^k) = \sum_{\gamma\in\{0,..,p-1\}} \left(\sum_{\alpha\in\mathbb{Z}_p} r(\omega^\alpha)\chi_A(\omega^{-\alpha\gamma})\right)\chi_A(\omega^{k\gamma})$$

is well-defined, for all $r \in \mathcal{C}[\mathcal{Z}_p]((A,*),(B,*))$, and is itself an arrow of $\mathcal{C}[\mathcal{Z}_p]((A,*),(B,*))$. First note that by the definition of arrows in a Cauchy product, the family $\{r(\omega^\alpha)\}_{\alpha\in\mathbb{Z}_p}$ is summable. Also, by Proposition 3.6 and the strong distributivity property, the families $\{\chi_A(\omega^{-\alpha'\gamma})\}_{(\alpha',\gamma)\in\mathbb{Z}_p\times\mathbb{Z}_p}$ and $\{\chi_A(\omega^{k\alpha''})\}_{(k,\alpha'')\in\{0,..,p-1\}\times\mathbb{Z}_p}$ are summable.

Therefore, by the strong distributivity property, the family

$$\{r(\omega^\alpha)\chi_A(\omega^{-\alpha'\gamma})\chi_A(\omega^{k\alpha''})\}_{(\alpha,\alpha',\gamma,k,\alpha'')\in\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p\times\{0,..,p-1\}\times\mathbb{Z}_p}$$

is summable, and so, by the summable subfamilies property, the subfamily given by restricting the index set to the subset where $\alpha = \alpha' = \alpha''$, is summable. Therefore, the sum

$$\sum_{(\gamma,\alpha,k)\in\mathbb{Z}_p\times\mathbb{Z}_p\times\{0,..,p-1\}} r(\omega^\alpha)\chi_A(\omega^{-\alpha\gamma})\chi_A(\omega^{k\gamma})$$

exists. By the weak partition-associativity axiom,

$$\sum_{k\in\{0,..,p-1\}} \left(\sum_{\gamma\in\mathbb{Z}_p}\left(\sum_{\alpha\in\mathbb{Z}_p} r(\omega^\alpha)\chi_A(\omega^{-\alpha\gamma})\right)\chi_A(\omega^{k\gamma})\right)$$

exists, and from the definition of $\mathfrak{G}$, this is equal to $\sum_{k\in K}\mathfrak{G}(\mathfrak{F}(r))$. Therefore, by definition of arrows in a Cauchy product, $\mathfrak{G}(\mathfrak{F}(r))$ is an arrow of $\mathcal{C}[\mathcal{Z}_p]((A,*),(B,*))$.

By appealing to Proposition 3.4 and the functoriality of χ_A, we may write this as

$$\mathfrak{G}(\mathfrak{F}(r))(\omega^k) = \sum_{\alpha\in\mathbb{Z}_p} r(\omega^\alpha)\left(\sum_{\gamma\in\{0,..,p-1\}} \chi_A\left(\omega^{(k-\alpha)\gamma}\right)\right)$$

However, by the orthogonality property (Proposition 7.6 above),

$$\sum_{\gamma\in\{0,..,p-1\}} \chi_A\left(\omega^{(k-\alpha)\gamma}\right)\begin{cases} \sum_{i=0}^{p-1} 1_A & k=\alpha \\ 0_{AA} & k\neq\alpha \end{cases}$$

Therefore

$$\mathfrak{G}(\mathfrak{F}(r))(\omega^k) = r(\omega^k)\sum_{i=0}^{p-1} 1_A = \sum_{i=0}^{p-1} r(\omega^k)$$

□

Corollary 7.9. The map $\mathfrak{F} : \mathcal{C}[\mathcal{Z}_p] \rightarrow \Delta^p\mathcal{C}$ of Theorem 7.5 is injective when the identity

$$\sum_{i=1}^{p} f \;=\; \sum_{j=1}^{p} g \quad\Rightarrow\quad f=g$$

holds for all $f, g \in Arr(\mathcal{C})$ (note that these sums exist, by the existence of a natural representation of $\mathcal{Z}_p$, and Proposition 3.6).

The above identity trivially holds when, at each $A \in Ob(\mathcal{C})$, there exists an arrow $({}^{1_A}\!/_p) \in \mathcal{C}(A, A)$ satisfying $\sum_{i=1}^{p} ({}^{1_A}\!/_p) = 1_A$. However, we cannot assume this property in arbitrary PCM-categories — even those with a natural representation of $\mathcal{Z}_p$.

8. Conclusions

We have demonstrated that the monoid-semiring construction can be placed within a significantly more general categorical setting. This has required an axiomatisation of summation that differs significantly from usual categorical axiomatisations of summation — the principle difference being that the positivity property is no longer imposed, and we are forced to rely on a strong form of distributivity for the required expressive power.

This allows us significantly more freedom in the structures we may define. In particular, it is possible to define and describe discrete Fourier transforms as functors of categories, rather than as homomorphisms of rings. However, in this more abstract setting the conditions required for the existence of a Fourier transform are, perhaps not unexpectedly, more subtle than those required in the usual setting. Work continues on a general theory of 'characters of categories', based on the simple prototype given in this paper.

9. Future directions

As well as the program outlined above, work continues in three interdependent directions:

9.1. *Categorical enrichment*

A natural question about this paper is whether the notion of 'PCM-category' is in fact an example of categorical enrichment (as in (Kelly 1982)) over the category **PCM**? Enrichment requires either a monoidal, or a closed, (or monoidal closed) structure. It has recently been demonstrated by the author and P. Scott (Ottawa) that **PCM** is a closed category in the sense of (Laplaza 1977), and a monoidal tensor adjoint to the closed structure has been given explicitly by T. Porter (Wales). This monoidal tensor appears to exhibit a universal property for a suitable notion of bilinear maps of PCMs (similar to the constructions of (Bahamonde 1985), in the context of Partially Additive Monoids). It is expected that a category enriched over this monoidal closed category is exactly a 'PCM-category', as defined in Definition 3.1. This is the subject of a forthcoming paper.

9.2. *The Cauchy product and monoidal structures*

Although we have demonstrated that the Cauchy product is a bifunctor, with interesting embedding properties, we have not yet considered the case where either the base category, or the index category (or both) have a monoidal tensor. This requires studying the monoidal structure of **PCM** (as above) in order to describe what it means for a PCM-category to have a monoidal tensor. This is undoubtedly an interesting route to explore. It is also important for questions relating to categorical Fourier transforms – observe that the approach taken in (Britten, Lemire 1981) to the characterisation of the commutative case is dependent on the tensor product and direct sum in the category of abelian groups. This question is, of course, also central to Shor's hidden subgroup algorithm (Shor 1999) and various generalisations (Lomont 2004).

9.3. *PCM-categories, algebraic program semantics, and axiomatisations of summation*

From the beginning, the notion of a PCM-category was intended to generalise certain categories with a suitable notion of summation on their hom-sets, used in algebraic program semantics. When a suitable treatment of monoidal tensors (as above) has been given we may consider how much of the standard theory (such as matrix representations, the Elgot dagger, the particle-style trace, &c.) may be reproduced in this more general setting.

Alternatively, it would be an interesting exercise to see how little structure is needed to specify either absolute convergence, or similar properties, within PCM categories (i.e. not simply within PCMs. Much of the expressive power comes from the strong distributivity axiom relating composition and summation). Notably, it seems reasonable to expect a single, seemingly trivial, assumption will lead to the summability of *all* finite families in a PCM-category.

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10. Appendix: PCMs and PCM-categories

We consider various examples of both PCMs and PCM-categories, as defined in Definitions 2.1 and 3.1 respectively. We also compare with other axiomatisations of summation from the field of algebraic program semantics.

10.1. *Examples of PCMs from algebraic program semantics*

Both Σ-*monoids*, and *partially additive monoids*, as introduced in (Manes, Benson 1985; Manes, Arbib 1986) and used in (Haghverdi 2000; Abramsky et. al. 2002; Haghverdi, Scott 2006), may be given as special cases of PCMs:

Definition 10.1. *(Σ-monoids, Partially additive monoids)*
A PCM (M, Σ) is called a **Σ-monoid** when it satisfies the following additional axiom:

— The **(full) Partition-Associativity Axiom.** Let $\{x_i\}_{i\in I}$ be a countably indexed family, and let $\{I_j\}_{j\in J}$ be a countable partition of I. Then $\{x_i\}_{i\in I}$ is summable if and only if $\{x_i\}_{i\in I_j}$ is summable for every $j \in J$, and $\{\sum_{i\in I_j} x_i\}_{j\in J}$ is summable, in which case
$$\sum_{i\in I} x_i = \sum_{j\in J}\left(\sum_{i\in I_j} x_i\right)$$

Note that this is a special case of the *weak partition-associativity axiom*, with a two-way, instead of a one-way, implication.

A Σ-monoid is called a **Partially Additive Monoid** (PAM) when it satisfies the following additional axiom:

— The **Limit Axiom.** Given $\{x_i\}_{i\in I}$, a countably indexed family where$\{x_i\}_{i\in F}$ is summable for every finite $F \subseteq I$, then $\{x_i\}_{i\in I}$ is summable.

The following are Partially Additive Monoids, and are therefore examples of PCMs:

— *Partial functions, with the usual summation*
An indexed family of partial functions $\{f_i : X \rightarrow Y\}_{i\in I}$ is **summable** exactly when $dom(f_i) \cap dom(f_j) = \emptyset$ for all $i \neq j$. The **sum** is given by:
$\left(\sum_{i\in I} f_i\right)(x) = \begin{cases} f_i(x) & x \in dom(f_i) \\ \text{undefined} & \text{otherwise} \end{cases}$

— *Relations, with set-theoretic union*
Any indexed family of relations $\{R_i : X \rightarrow Y\}_{i\in I}$ is **summable**, and the **sum** is simply set-theoretic union.

— *Partial injective functions*
The following distinct summations both give a PAM structure to hom-sets of partial injective functions:

  – *The disjointness summation* An indexed family of partial functions $\{f_i : X \rightarrow Y\}_{i\in I}$ is **disjointness-summable** exactly when $dom(f_i) \cap dom(f_j) = \emptyset$ for all $i \neq j$. The **sum** is given by:
  $\left(\sum_{i\in I} f_i\right)(x) = \begin{cases} f_i(x) & x \in dom(f_i) \\ \text{undefined} & \text{otherwise} \end{cases}$
  – *The overlap summation* An indexed family of partial functions $\{f_i : X \rightarrow Y\}_{i\in I}$ is **overlap-summable** exactly when $x \in dom(f_i) \cap dom(f_j) \;\Rightarrow\; f_i(x) = f_j(x)$, for all $i, j \in I$. The **sum** is as given above.

The following example is not a partially additive monoid, but is a Σ-monoid, and thus also an example of a PCM:

— *Absolute convergence on positive cones*
We refer to (Selinger 2004) for categories of positive cones, and summation based on the usual summation of positive elements in vector space.

Taking as a general principle that our motivating examples (including group-rings, and complex formal power series that converge within the unit disk) should be specific examples of the general constructions introduced, both Σ-monoids, and Partially Additive Monoids have undesirable properties for our purposes. The *limit axiom* is clearly undesirable for any example based on real or complex numbers: all finite families of complex numbers are summable, but the same is certainly not true (as the limit axiom would imply) for arbitrary countably infinite families.

The full partition-associativity axiom is also undesirable for slightly more subtle reasons, as the following proposition (taken from (Manes, Arbib 1986)) demonstrates:

Proposition 10.2. Let (M,Σ) be a Σ-monoid, and let $X = \{x_i\}_{i\in I}$ be a summable family of M satisfying $\sum_{i\in I} x_i = 0$. Then $x_i = 0$ for all $i \in I$.

Proof. For some $i \in I$, we define $Y = \{x_j\}_{j\neq i\in I}$, so $x_i + \sum Y = 0 = \sum Y + x_i$ by weak partition associativity. Then by the full partition-associativity axiom,

$$\begin{aligned} x_i &= x_i + 0 + 0 + 0 + \dots \quad \text{by Proposition 2.4} \\ &= x_i + (\sum Y + x_i) + (\sum Y + x_i) + (\sum Y + x_i) + \dots \\ & \text{(by full partition associativity)} \\ &= (x_i + \sum Y) + (x_i + \sum Y) + (x_i + \sum Y) + \dots \\ &= 0 + 0 + \cdots \ = \ 0, \text{by Proposition 2.4} \end{aligned}$$

Hence $x_i = 0$. However, as i was chosen arbitrarily, $x_k = 0$ for all $k \in I$. □

10.2. *Other examples of PCMs*

The above proof of positivity does *not* apply to general PCMs, as it depends on the two-way implication in the (full) partition-associativity axiom. We give various examples of PCMs that need not be either Partial Additive Monoids or Sigma-monoids. Many of these are based on the theory of Cauchy sequences, and the structure of absolutely convergence – we refer to (Hobson 1957; Titchmarsh 1983) for the relevant background.

— *Absolute convergence of real or complex numbers* Absolute convergence of countable sums in the real or complex plane is a motivating example for the theory of PCMs and PCM-categories. It arises as a special case of absolute convergence in Hilbert spaces, as below:

— *Absolute convergence in Hilbert spaces* All finitely indexed families are summable, with the usual summation. A countably indexed family $\{\psi_i\}_{i\in\mathbb{N}}$ is **summable** exactly when the sequence $\{\sum_{i=1}^{n} \|\psi_i\|\}_{n\in\mathbb{N}}$ is a Cauchy sequence, in which case the **sum** is the limit of the Cauchy sequence $\{\sum_{n=1}^{N} \psi_n\}_{N\in\mathbb{N}}$, which exists by the completeness of Hilbert spaces. We refer to any introductory analysis book for a proof that this is a Cauchy sequence and the limit is independent of reordering. The weak partition-associativity axiom then follows from the property that every sub-sequence of a Cauchy sequence is itself a Cauchy sequence, and the independence of Cauchy sequences under reordering (Titchmarsh 1983).

— *The unit ball summation in Banach spaces* Let $\mathcal{B}$ be a Banach space, and denote the unit ball by $Ball(\mathcal{B}) = \{b \in \mathcal{B} : \|b\| \leq 1\}$. An indexed family $\{b_i\}_{i\in I}$ is summable exactly when $\sum_{i\in I} \|b_i\| \leq 1$, in which case its sum is the usual Banach space summation.

— *Any abelian monoid* Given an abelian monoid $(M, +, 0_M)$, then the following are distinct PCM-structures:

- *The finite families summation* Given a finitely indexed set $\{m_i\}_{i\in I}$, with $|I| \cong \{1, \ldots, n\}$, then $\Sigma_{i\in I} m_i = m_1 + m_2 + \ldots + m_n$, when $n > 1$. When $|I| = 1$, then $\Sigma_{i\in I} m_i = m_i$, and when $I = \emptyset$, then $\Sigma_{i\in I} m_i = 0_M$.
- *The K-bounded summation* All finite families with at most K non-zero elements are summable, in which case the summation is as above.

It is almost immediate from the commutativity and associativity of composition in M that these both satisfy the PCM axioms.

10.3. *Examples of PCM-categories*

A PCM-category is defined in Definition 3.1 to be a category $\mathcal{C}$ where each hom-sets has a specified PCM structure, together with the *strong distributivity* axiom that connects composition and summation. This states that, given summable families $\{g_j \in \mathcal{C}(Y, Z)\}_{j\in I}$ and $\{f_i \in \mathcal{C}(X, Y)\}_{i\in I}$, then $\{g_j f_i \in \mathcal{C}(X, Z)\}_{(j,i)\in\mathcal{C}(X,Z)}$ is summable and

$$\left(\sum_{j\in J} g_j\right)\left(\sum_{i\in I} f_i\right) = \sum_{(j,i)\in J\times I} g_j f_i$$

— *The real numbers, with multiplication and absolute convergence*
Given two absolutely convergent sums of reals, $\sum_{i\in I} r_i$ and $\sum_{j\in J} s_j$, then by definition of absolute convergence, $\sum_{(i,j)\in I\times J} r_i s_j$ exists, and $\left(\sum_{i\in I} r_i\right)\left(\sum_{j\in J} s_j\right) = \sum_{(i,j)\in I\times J} r_i s_j$. Therefore, $(\mathbb{R}, \times)$, with this indexed summation, is a one-object PCM-category.

— *Any ring, with the finite families summation*
Let $(R, \times, +)$ be a ring. Then $(R, \times)$ is a monoid, and hence a one-object category. Its unique homset (i.e. the elements of R) is an abelian monoid, $(R, +)$. Hence we may take the the finite families summation $\Sigma_{<\infty}$ of Section 10.2 above to get the PCM $(R, \Sigma_{<\infty})$. Given two finite (i.e. summable) families, $\{s_j\}_{j\in J}$ and $\{r_i\}_{i\in I}$, then the family $\{s_j r_i\}_{(j,i)\in J\times I}$ is finite, and hence summable. Given the summability of the required families, the identity

$$\left(\sum_{j\in I} s_j\right)\left(\sum_{i\in I} r_i\right) = \sum_{(j,i)\in J\times I} s_j r_i$$

is then straightforward from the definition of summation in terms of the addition in the ring $(R, \times, +)$.

A non-example: Given a ring $R, \times, +)$, the PCM structure on the abelian monoid $(R, +)$ is not unique. Section 10.2 describes the family of K-bounded summations $\Sigma_{\leq K}$ where a family is summable exactly when it has no more than K non-zero elements. However, for $K > 1$, the K-bounded summation does *not* in general make $(R, \times, \Sigma_{\leq K})$ a one-object PCM category. Let us assume that R has no zero-divisors, and consider two summable families of K non-zero elements, $\{s_j\}_{j=1}^K$ and $\{r_i\}_{i=1}^K$. Then the strong distributivity law does not hold, since $\{s_j r_i\}_{(j,i)\in\{1,..,K\}^2}$ is not a summable family, as it consists of $K^2 > K$ non-zero elements.

We now demonstrate that there is a whole class of examples to be found within the field of algebraic program semantics. The proofs that these are PCM-categories arises from the following straightforward result:

Proposition 10.3. Let $\mathcal{C}$ be a category, together with, for all $X, Y \in Ob(\mathcal{C})$ a function $\Sigma^{(X,Y)}$ from indexed families over $\mathcal{C}(X, Y)$ to $\mathcal{C}(X, Y)$ such that:

1 $\left(\mathcal{C}(X, Y), \Sigma^{(X,Y)}\right)$ is a PCM satisfying the additional *full partition-associativity axiom* of Definition 10.1.

2 The usual left and right distributivity conditions as satisfied: that is, given a summable family $\{g_i \in \mathcal{C}(B, C)\}_{i\in I}$ and arbitrary $f \in \mathcal{C}(A, B)$ and $h \in \mathcal{C}(C, D)$, then the families $\{hg_i \in \mathcal{C}(B, D)\}_{i\in I}$ and $\{g_i f \in \mathcal{C}(A, C)\}_{i\in I}$ are summable and

$$h\left(\sum_{i\in I} g_i\right) = \sum_{i\in I}(hg_i) \quad \text{and} \quad \left(\sum_{i\in I} g_i\right) f = \sum_{i\in I}(g_i f)$$

Then $\mathcal{C}, \Sigma^{(\_,\_)}$ satisfies the strong distributivity condition of Definition 3.1, and thus is a PCM-category.

Proof. Let us write $f = \sum_{i\in I} f_i$ and $g = \sum_{j\in J} g_j$. Then, By left-distributivity, and the existence of $\sum_{i\in I} f_i$, we deduce $gf = \sum_{i\in I} gf_i$. However, $g = \sum_{j\in J} g_j$. Therefore, by full partition-associativity,

$$gf = \sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right)$$

Using the right distributivity law and full partition-asociativity,

$$gf = \sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right)$$

Again by full partition-associativity, and Proposition 2.3, we may replace the doubly-indexed sum by a single indexed sum, giving $gf = \sum_{(j,i)\in J\times I} g_j f_i$ and hence

$$gf = \left(\sum_{j\in J} g_j\right)\left(\sum_{i\in I} f_i\right) = \sum_{(j,i)\in J\times I} g_j f_i$$

Therefore $\mathcal{C}$ satisfies strong distributivity, as required. □

Note that the converse is not true: weak partition-associativity, together with the strong distributivity law, does not, in general, imply the full partition-associativity axiom. This is clear from the failure of positivity in many of the examples given.

Corollary 10.4. The Partially Additive Categories (PACs) of (Manes, Arbib 1986) are PCM-categories, as are the Unique Decomposition Categories (UDCs) of (Haghverdi 2000; Abramsky et. al. 2002; Haghverdi, Scott 2006).

Proof. The definitions of PACs and UDCs include, as axioms, 1.-2. above. □