Propositional Calculus

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Propositions

*A **proposition** is some statement such as*

1. 4 *is a prime number*
2. $3 + 3 = 6$
3. 2 *is an even integer, but* 3 *is not*
4. *my program always halts, if it runs for long enough*

fundamental property

- *either true or false*
- *but not both*

truth value

Compound propositions

fundamental property of a compound proposition

- *truth value determined by subpropositions and connectives*

propositional variable

- *denotes arbitrary proposition with unspecified truth value*
- p, q, r, p_1, q_2, r_3
- H : *"my program always halts, if it runs for long enough"*

Connectives

$\neg$	*negation*	***not***
$\wedge$	*conjunction*	***and***
$\vee$	*disjunction*	***or***
$\Rightarrow$	*implication*	***implies***
$\Leftrightarrow$	*equivalence*	***if and only if***
		is equivalent to

order of precedence

$$\neg p \wedge q \vee r \Leftrightarrow q \Rightarrow p \wedge r$$

is equivalent to

$$(((\neg p) \wedge q) \vee r) \Leftrightarrow (q \Rightarrow (p \wedge r))$$

Examples

- $\neg$ *Jaffa cakes are biscuits*
- $\pi < 4 \wedge \pi < 5$
- $\pi < 4 \Rightarrow \pi < 5$
- *Leo is twenty-something* $\Rightarrow$ *Leo is under thirty*
- *Leo is twenty-something* $\Leftrightarrow$ *Leo is under thirty*

Conjunction

- *any two propositions may be connected together using the word "and" to form a compound proposition called the **conjunction** of the original propositions*
- *"Roses are red **and** violets are blue"*
- *conjunction written symbolically as* $p \wedge q$
- *read "p and q"*
- *true exactly when both p and q are true*
- *false otherwise*

Disjunction

- *any two propositions may be connected together using the word "or" to form a compound proposition called the **disjunction** of the original propositions*
- *"The bridge is safe, or Jim is a poor engineer"*
- *disjunction written symbolically as* $p \vee q$
- *read "p or q"*
- *true exactly either when p is true, or when q is true*
- *false otherwise*
- ***inclusive** or*

Example: formalising English propositions

- *lawyers use the term "and/or" to denote inclusive or*
- *what if we wanted to describe **exclusive or**?*
- ***"you may have a beer or you may have a whisky"***
- *"you may have a beer or you may have a whisky, but not both"*
- *B or W, but not both*
- *B or W, but not both B and W*
- *(B or W) $\wedge$ (not both B and W).*
- *($B \vee W$) $\wedge$ (not both B and W).*
- *($B \vee W$) $\wedge$ $\neg$ (both B and W).*
- $(B \vee W) \wedge \neg(B \wedge W)$

Negation

- given any proposition p, another proposition called the **negation** of p may be formed by is using the unary connective "not"
- "It is not the case that roses are red and violets are blue"
- written symbolically as $\neg p$
- read "not p"
- true exactly when p is false
- "$\neg($roses are red $\wedge$ violets are blue$)$"

Implication

- "If p, then q"
- "If I eat Jaffa cakes, then I shall get fat"
- "I eat Jaffa cakes $\Rightarrow$ I get fat"
- **antecedent** $\Rightarrow$ **consequent**
- property: anything follows from a contradiction
- "If Napoleon is German, then I'm a Dutchman"
- "Napoleon is German $\Rightarrow$ I'm a Dutchman"
- this has truth value **true**
- **but I'm not Dutch!**

Implication as an order on truth values

- "false" is a very strong constraint, "true" is very weak
- $p \Rightarrow q$: "p implies q"
- "p is stronger than (or equal to) q"
- false is stronger than true
- true is weaker than false
- each truth value is as strong as itself
- ordering:

true ⇒ true	true
true ⇒ false	false
false ⇒ true	true
false ⇒ false	true

Example: formalising English propositions

"If the wind is not too strong and the thunderstorm has finished, then we can sail the boat"

$$\neg W \wedge \neg T \Rightarrow S$$

"If you are right, then I apologise, but I think that you are wrong"

$$(R \Rightarrow A) \wedge W$$

"You will find him, if you look under the desk"

$$D \Rightarrow F$$

Necessary and sufficient conditions

- *"You may enter only if you have a ticket."*
- *clearly an implication, but which way round?*
- *having a ticket is a* ***necessary*** *condition to enter*
- *may not be* ***sufficient***
- $p \Rightarrow q$
  - *"p is a sufficient condition for q"*
  - *"q is a necessary condition for p"*
- *"If you enter, then (at least) you have a ticket"*

- *formalisation*

  "p only if q" $\quad p \Rightarrow q$

  "p if q" $\quad q \Rightarrow p$

Converse and contrapositive

- ***converse*** *of* $p \Rightarrow q$
  - $q \Rightarrow p$
- ***contrapositive*** *of* $p \Rightarrow q$
  - $\neg q \Rightarrow \neg p$

Equivalence

- $p \Leftrightarrow q$
- *read as "p is equivalent to q"*
- *or "p if and only if q"*
- *abbreviated to "p iff q"*
- *"p is a necessary and sufficient condition for q*
- *true if and only if p and q have the same truth value*

Example

- *"a pass in FMS is equivalent to a pass in RQE"*
- $F \Leftrightarrow M$
- *"if and only if the time is right, will the revolution succeed"*
- $T \Leftrightarrow R$

Equivalence as a biconditional

- *"p if and only if q"*
- *"p if q and p only if q"*
- *"You will pass the MSc if and only if you pass the FMS assessment"*
- *"You will pass the MSc if you pass the FMS assessment, and you will pass the MSc only if you pass the FMS assessment"*
- $(A \Rightarrow M) \wedge (M \Rightarrow A)$
- $A \Leftrightarrow M$

Propositions and Truth Tables

- *repetitive use of logical connectives constructs more and more complicated compound propositions*
- *truth value depends on truth values of constituent parts, combined in various ways*
- ***function of component truth values***
- *truth tables evaluate truth value of compound propositions*

Truth tables

p	q	$p \wedge q$
t	t	t
t	f	f
f	t	f
f	f	f

p	q	$p \vee q$
t	t	t
t	f	t
f	t	t
f	f	f

p	q	$p \Rightarrow q$
t	t	t
t	f	f
f	t	t
f	f	t

p	q	$p \Leftrightarrow q$
t	t	t
t	f	f
f	t	f
f	f	t

p	$\neg p$
t	f
f	t

Using truth tables

- *example:* $\neg(p \wedge \neg q)$
- *list all propositional variables*
- *list all possible situations for propositional variables*
- 2^k *combinations, for* k *variables*
- *tabulate result in each situation*

Example (method 1)

build up results for subexpressions

$\neg(p \wedge \neg q)$

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
t	t	f	f	t
t	f	t	t	f
f	t	f	f	t
f	f	t	f	t

Example (method 2)

develop results in place

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
t	t					
t	f					
f	t					
f	f					

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
t	t		t			
t	f		t			
f	t		f			
f	f		f			

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
t	t		t			t
t	f		t			f
f	t		f			t
f	f		f			f

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
t	t		t		f	t
t	f		t		t	f
f	t		f		f	t
f	f		f		t	f

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
t	t		t	f	f	t
t	f		t	t	t	f
f	t		f	f	f	t
f	f		f	f	t	f

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
t	t	t	t	f	f	t
t	f	f	t	t	t	f
f	t	t	f	f	f	t
f	f	t	f	f	t	f

Final result

p	q	$\neg(p \wedge \neg q)$
t	t	t
t	f	f
f	t	t
f	f	t

this is the same truth table as that for $p \Rightarrow q$

thus, we have proved the equivalence of two propositions

Tautologies and Contradictions

- ***tautologies:*** *propositions that are true everywhere*
- ***contradictions:*** *propositions that false everywhere*
- *negation of a contradiction is a tautology*
- *and* ***vice versa***
- ***contingencies:*** *neither tautologies nor contradictions*

Example: tautology

tautology: "every proposition is either true or false"

$p \vee \neg p$

p	p	$\vee$	$\neg$	p
t				
f				

p	p	$\vee$	$\neg$	p
t	t			
f	f			

p	p	$\vee$	$\neg$	p
t	t			t
f	f			f

p	p	$\vee$	$\neg$	p
t	t		f	t
f	f		t	f

p	p	$\vee$	$\neg$	p
t	t	t	f	t
f	f	t	t	f

p	p	$\vee$	$\neg$	p
t	t	$\boxed{t}$	f	t
f	f	$\boxed{t}$	t	f

Example: contradiction

contradiction: "a proposition can be both true and false"

$p \wedge \neg p$

p	p	$\wedge$	$\neg$	p
t				
f				

p	p	$\land$	$\neg$	p
t	t			
f	f			

p	p	$\land$	$\neg$	p
t	t			t
f	f			f

p	p	$\land$	$\neg$	p
t	t		f	t
f	f		t	f

p	p	$\land$	$\neg$	p
t	t	f	f	t
f	f	f	t	f

p	p	$\wedge$	$\neg$	p
t	t	**f**	f	t
f	f	**f**	t	f

Equivalences

- *p and q are said to be **logically equivalent** if the proposition $p \Leftrightarrow q$ is a tautology*
- *p and q have the same truth table*
- *if two propositions are logically equivalent, then one may be substituted for the other in any proposition in which they occur*
- *p is equivalent to $p \vee p$*
- *so $p \vee q$ is equivalent to $(p \vee p) \vee q$*

Identities

1.	$p \Leftrightarrow p \vee p$	*idempotence of* $\vee$
2.	$p \Leftrightarrow p \wedge p$	*idempotence of* $\wedge$
3.	$p \vee q \Leftrightarrow q \vee p$	*commutativity of* $\vee$
4.	$p \wedge q \Leftrightarrow q \wedge p$	*commutativity of* $\wedge$
5.	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	*associativity of* $\vee$
6.	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	*associativity of* $\wedge$
7.	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	*De Morgan's Law (1)*
8.	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	*De Morgan's Law (2)*
9.	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	*distributivity of* $\wedge$ *over* $\vee$
10.	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	*distributivity of* $\vee$ *over* $\wedge$

Identities

11.	$p \vee true \Leftrightarrow true$	*zero for* $\vee$
12.	$p \wedge true \Leftrightarrow p$	*unit for* $\wedge$
13.	$p \vee false \Leftrightarrow p$	*unit for* $\vee$
14.	$p \wedge false \Leftrightarrow false$	*zero for* $\wedge$
15.	$p \vee \neg p \Leftrightarrow true$	*excluded middle*
16.	$p \wedge \neg p \Leftrightarrow false$	*contradiction*
17.	$p \Leftrightarrow \neg\neg p$	*double negation*
18.	$(p \Rightarrow q) \Leftrightarrow \neg p \vee q$	*implication*
19.	$(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$	*equivalence*
20.	$(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow (q \Rightarrow r))$	*exportation*
21.	$(p \Rightarrow q) \wedge (p \Rightarrow \neg q) \Leftrightarrow \neg p$	*absurdity*
22.	$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$	*contraposition*

Example: using identities to simplify

$(p \Rightarrow q) \vee (p \Rightarrow r) \Rightarrow (q \vee r)$

$\Leftrightarrow$ *by implication, twice*

$(\neg p \vee q) \vee (\neg p \vee r) \Rightarrow (q \vee r)$

$\Leftrightarrow$ *by commutativity, associativity, and idempotence of* $\vee$

$\neg p \vee (q \vee r) \Rightarrow (q \vee r)$

$\Leftrightarrow$ *by implication*

$\neg(\neg p \vee (q \vee r)) \vee (q \vee r)$

$\Leftrightarrow$ *by De Morgan's Law (1)*

$(\neg\neg p \wedge \neg(q \vee r)) \vee (q \vee r)$

$\Leftrightarrow$ *by double negation*

$(p \wedge \neg(q \vee r)) \vee (q \vee r)$

$\Leftrightarrow$ *by commutativity of* $\vee$

$(q \vee r) \vee (p \wedge \neg(q \vee r))$

$\Leftrightarrow$ *by distributivity of* $\vee$ *over* $\wedge$

$((q \vee r) \vee p) \wedge ((q \vee r) \vee \neg(q \vee r))$

$\Leftrightarrow$ *by excluded middle*

$((q \vee r) \vee p) \wedge \mathit{true}$

$\Leftrightarrow$ *by unit for* $\wedge$

$(q \vee r) \vee p$

$\Leftrightarrow$ *by commutativity*

$p \vee (q \vee r)$

Inequalities

1.	$p \Rightarrow p \vee q$	*addition*
2.	$p \wedge q \Rightarrow p$	*simplification*
3.	$p \wedge (p \Rightarrow q) \Rightarrow q$	***modus ponens***
4.	$(p \Rightarrow q) \wedge \neg q \Rightarrow \neg p$	***modus tollens***
5.	$\neg p \wedge (p \vee q) \Rightarrow q$	*disjunctive syllogism*
6.	$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$	*hypothetical syllogism*
7.	$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$	*transitivity of* $\Rightarrow$
8.	$(p \Rightarrow q) \wedge (r \Rightarrow s) \Rightarrow ((p \wedge r) \Rightarrow (q \wedge s))$	*coupling*
9.	$(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$	*transitivity of* $\Leftrightarrow$

Example

- *Gerald was captured by an alien*
- *he was promised his freedom if he could determine with a single "yes or no" question the colour of the alien's spaceship*
- *he knew the spaceship was either black or white*
- *unfortunately, there are two kinds of aliens: liars and truth tellers*
- *fortunately, the man was well-educated by DMR*
- *what question did he ask?*
- T*: the man is a truth-teller*
- W*: the spaceship is white*

A bad guess

Q_1: *is the spaceship white?*

$Q_1 \Leftrightarrow (T \wedge W) \vee (\neg\, T \wedge \neg\, W)$

T	W	$(T \wedge W)$	$\vee$	$(\neg\, T \wedge \neg\, W)$
t	t			
t	f			
f	t			
f	f			

T	W	$(T \wedge W)$	$\vee$	$(\neg\, T \wedge \neg\, W)$
t	t	t		
t	f	f		
f	t	f		
f	f	f		

T	W	$(T \wedge W)$	$\vee$	$(\neg\, T \wedge \neg\, W)$
t	t	t		f
t	f	f		f
f	t	f		t
f	f	f		t

T	W	$(T \wedge W)$	$\vee$	$(\neg T$	$\wedge$	$\neg W)$
t	t	t		f		f
t	f	f		f		t
f	t	f		t		f
f	f	f		t		t

T	W	$(T \wedge W)$	$\vee$	$(\neg T$	$\wedge$	$\neg W)$
t	t	t		f	f	f
t	f	f		f	f	t
f	t	f		t	f	f
f	f	f		t	t	t

T	W	$(T \wedge W)$	$\vee$	$(\neg T$	$\wedge$	$\neg W)$
t	t	t	**t**	f	f	f
t	f	f	**f**	f	f	t
f	t	f	**f**	t	f	f
f	f	f	**t**	t	t	t

So what happened?

the alien ate Gerald

A better guess

Q_2 : *"Is it true that either you tell the truth and the spaceship is white, or that you lie and the spaceship is black?"*

$Q_2 \Leftrightarrow (T \Leftrightarrow (T \wedge W) \vee (\neg\, T \wedge \neg\, W))$

truth-teller $(true \Leftrightarrow (true \wedge W) \vee (\neg\, true \wedge \neg\, W))$

liar $(false \Leftrightarrow (false \wedge W) \vee (\neg\, false \wedge \neg\, W))$

T	W	$(T \Leftrightarrow (T \wedge W) \vee (\neg\, T \wedge \neg\, W))$
t	t	
t	f	
f	t	
f	f	

T	W	$(T \Leftrightarrow$	$(T \wedge W) \vee$	$(\neg\, T \wedge \neg\, W))$
t	t		t	
t	f		f	
f	t		f	
f	f		f	

T	W	$(T \Leftrightarrow$	$(T \wedge W) \vee$	$(\neg\, T \wedge \neg\, W))$
t	t		t	f
t	f		f	f
f	t		f	t
f	f		f	t

T	W	$(T$	$\Leftrightarrow$	$(T$	$\wedge$	$W)$	$\vee$	$(\neg$	T	$\wedge$	$\neg$	$W))$
t	t				t			f			f	
t	f				f			f			t	
f	t				f			t			f	
f	f				f			t			t	

T	W	$(T$	$\Leftrightarrow$	$(T$	$\wedge$	$W)$	$\vee$	$(\neg$	T	$\wedge$	$\neg$	$W))$
t	t				t		t	f		f	f	
t	f				f		f	f		f	t	
f	t				f		f	t		f	f	
f	f				f		t	t		t	t	

T	W	$(T$	$\Leftrightarrow$	$(T$	$\wedge$	$W)$	$\vee$	$(\neg$	T	$\wedge$	$\neg$	$W))$
t	t				t			f		f	f	
t	f				f			f		f	t	
f	t				f			t		f	f	
f	f				f			t		t	t	

T	W	$(T$	$\Leftrightarrow$	$(T$	$\wedge$	$W)$	$\vee$	$(\neg$	T	$\wedge$	$\neg$	$W))$
t	t		**t**		t		t	f		f	f	
t	f		**f**		f		f	f		f	t	
f	t		**t**		f		f	t		f	f	
f	f		**f**		f		t	t		t	t	

So what happened?

- *this question enabled Gerald to determine the colour correctly*
- *since an answer* ***yes*** *meant the spaceship was white*
- *and an answer* ***no*** *meant that it was black*
- *unfortunately, the alien thought that it was just a lucky guess*
- ***so he ate Gerald anyway!***

Modus ponens

- *suppose* p *and* $p \Rightarrow q$ *are both true*
- q *must also be true*
- *recall the truth table for implication:*

p	q	$p \Rightarrow q$
t	t	t
t	f	f
f	t	t
f	f	t

Arguments

- ***argument:*** *an assertion that a given set of propositions* $p_1, p_2, \ldots, p_n$*, called the premises, entails another proposition* q*, called the conclusion*

$$p_1, p_2, \ldots, p_n \vdash q$$

- *the argument* $p_1, p_2, \ldots, p_n \vdash q$ *is valid if* q *is true whenever all the premises* $p_1, p_2, \ldots, p_n$ *are true*
- ***fallacy:*** *an argument which isn't valid*

Example (modus morons)

the following argument is a fallacy

$$p \Rightarrow q, q \vdash p$$

proof of this follows directly from the truth table for implication

Example

Theorem: *if* $x^2 - 3x + 2 < 0$, *then* $x > 0$

Proof 1:

assume that $x^2 - 3x + 2 < 0$

$$3x > x^2 + 2 \geq 2 \qquad \textit{since } x^2 \geq 0$$

hence

$$x > \tfrac{2}{3} > 0$$

it follows that if $x^2 - 3x + 2 < 0$, *then* $x > 0$

Proof 2:

contrapositive: "if $x \leq 0$, *then* $x^2 - 3x + 2 \geq 0$*"*

assume $x \leq 0$

$$x - 1 \leq 0 \quad \textit{and} \quad x - 2 \leq 0$$

hence

$$x^2 - 3x + 2 = (x-1)(x-2) \geq 0$$

it follows that "if $x \leq 0$, *then* $x^2 - 3x + 2 \geq 0$*"*

hence "if $x^2 - 3x + 2 < 0$, *then* $x > 0$*"*

Proof 3:

assume "$x^2 - 3x + 2 < 0$*"*

assume "$x \leq 0$*"*

$$x^2 < 3x - 2 \leq -2 < 0$$

this is a contradiction

hence "if $x^2 - 3x + 2 < 0$, *then* $x > 0$*"*

Increasing formality

$x^2 - 3x + 2 < 0$

$\Leftrightarrow$ *by rearranging*

$3x > x^2 + 2$

$\Leftrightarrow$ *since* $x^2 \geq 0$, *we have that* $x^2 + 2 \geq 0 + 2$

$3x > x^2 + 2 \wedge x^2 + 2 \geq 2$

$\Leftrightarrow$ *by the meaning of* $\geq$

$3x > x^2 + 2 \wedge (x^2 + 2 > 2 \vee x^2 + 2 = 2)$

$\Leftrightarrow$ *by distributivity of* $\wedge$ *over* $\vee$

$(3x > x^2 + 2 \wedge x^2 + 2 > 2) \vee (3x > x^2 + 2 \wedge x^2 + 2 = 2)$

$\Rightarrow$ *by transitivity of* $>$

$(3x > 2) \vee (3x > x^2 + 2 \wedge x^2 + 2 = 2)$

$\Rightarrow$ *using the equality in the second disjunct*

$(3x > 2) \vee (3x > 2)$

$\Leftrightarrow$ *by idempotence of* $\vee$

$3x > 2$

$\Leftrightarrow$ *dividing both sides by* 3

$x > \frac{2}{3}$

$\Rightarrow$ *since* $\frac{2}{3} > 0$*, and* $>$ *is transitive*

$x > 0$

The Boolean satisfiability problem (SAT)

- *decision problem*
- *Boolean expression using AND, OR, NOT, variables, and parentheses*
- *is there some assignment of TRUE and FALSE values to the variables that will make the entire expression true?*
- *proposition is satisfiable if truth values can be assigned to variables to make the proposition true*
- *SAT is centrally important in theoretical computer science, algorithmics, artificial intelligence, hardware design, and software verification*
- *Cook's theorem: SAT NP-complete*

Propositional Calculus Problems

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1. *Let p be "it's cold" and let q be "it's raining". Give a simple verbal sentence which describes each of the following propositions:*
   (a) $\neg p$
   (b) $p \wedge q$
   (c) $p \vee q$
   (d) $q \vee \neg p$
   (e) $\neg p \wedge \neg q$
   (f) $\neg\neg q$

2. *Let p be "she's tall" and let q be "she's beautiful". Write propositions that symbolise the following:*
   (a) *She's tall and beautiful.*
   (b) *She's tall but not beautiful.*
   (c) *It's false that she's short or beautiful.*
   (d) *She's neither tall nor beautiful.*
   (e) *It isn't true that she's short or not beautiful.*

3. *Translate into symbols the following compound statements:*
   (a) *We'll win the election, provided that Brown is elected leader of the party.*
   (b) *If Brown isn't elected leader of the party, then either Straw or Johnson will leave the cabinet, and we'll lose the election.*
   (c) *If x is a rational number and y is an integer, then z isn't real.*
   (d) *Either the murderer has left the country or somebody is hiding him.*
   (e) *If the murderer has not left the country, then somebody is hiding him.*
   (f) *The sum of two numbers is even if and only if either both numbers are even or both numbers are odd.*
   (g) *If y is an integer, then z isn't real, provided that x is a rational number.*

4. *Let p be the proposition "It's snowing." Let q be the proposition "I'll go to town." Let r be the proposition "I have time."*
   (a) *Using propositional connectives, write a proposition which symbolises each of the following:*
      i. *If it isn't snowing and I have time, then I'll go to town.*
      ii. *I'll go to town only if I have time.*
      iii. *It isn't snowing.*
      iv. *It's snowing, and I'll not go to town.*

   (b) *Write a sentence in English corresponding to each of the following propositions:*
      i. $q \Leftrightarrow r \wedge \neg p$
      ii. $r \wedge q$
      iii. $(q \Rightarrow r) \wedge (r \Rightarrow q)$
      iv. $\neg(r \vee q)$

5. *State the converse and contrapositive of each of the following:*
   (a) *If it rains, I'm not going.*
   (b) *I'll stay only if you go.*
   (c) *If you get 4lbs, you can bake the cake.*
   (d) *I can't complete the task if I don't get more help.*

6. *Let p, q, r denote the following statements:*
   - p: *Triangle ABC is isosceles;*
   - q: *Triangle ABC is equilateral;*
   - r: *Triangle ABC is equiangular.*

   Translate each of the following into an English sentence.
   (a) $q \Rightarrow p$
   (b) $\neg p \Rightarrow \neg q$
   (c) $q \Leftrightarrow r$
   (d) $p \wedge \neg q$
   (e) $r \Rightarrow p$

7. (a) How many rows are needed for the truth table of the following proposition?

$$p \vee \neg q \Leftrightarrow \neg r \wedge s \Rightarrow t$$

   (b) If $p_1, p_2, \ldots, p_n$ are propositional variables, and the compound statement p contains at least one occurrence of each propositional variable p_i, how many rows are needed in order to construct the truth table for p?

8. Find truth tables for the following propositions:
   (a) $\neg p \wedge q$
   (b) $\neg(p \vee q)$
   (c) $\neg(p \vee \neg q)$

9. Express each of the following statements in the form “$p \Rightarrow q$”.
   (a) Rain on Tuesday is a necessary condition for rain on Sunday.
   (b) If it rains on Tuesday, then it rains on Wednesday.
   (c) But it rains on Wednesday only if it rains on Friday.
   (d) Moreover, no rain on Monday means no rain on Friday.
   (e) Finally, rain on Monday is a sufficient condition for rain on Saturday.

   Given that it rains on Sunday, what can be said about Saturday’s weather?

10. Verify that the proposition $p \vee \neg(p \wedge q)$ is a tautology.

11. Verify that the proposition $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction.

12. Which of the following propositions are tautologies?
   (a) $p \Rightarrow (q \Rightarrow p)$
   (b) $q \vee r \Rightarrow (\neg r \Rightarrow q)$
   (c) $(p \wedge \neg q) \vee ((q \wedge \neg r) \vee (r \wedge \neg p))$
   (d) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge \neg q) \vee r)$

13. Show that the following pairs of propositions are logically equivalent.
   (a) $p \Rightarrow q, \quad \neg q \Rightarrow \neg p$
   (b) $(p \vee q) \wedge r, \quad (p \wedge r) \vee (q \wedge r)$
   (c) $\neg p \wedge \neg q \Rightarrow \neg r, \quad r \Rightarrow q \vee p$
   (d) $\neg p \vee q \Rightarrow r, \quad (p \wedge \neg q) \vee r$

14. Show that the proposition $(\neg p \Rightarrow q) \Rightarrow (p \Rightarrow \neg q)$ isn’t a tautology.

15. Show that the following argument is valid: $p \Rightarrow q, \neg q \vdash \neg p$.

16. Show that the following argument is valid: $p \Leftrightarrow q, q \vdash p$.

17. Show that the following argument is a fallacy: $p \Rightarrow q \vdash \neg p \Rightarrow \neg q$.

18. Test the validity of the following argument:

    If I study, then I’ll not fail mathematics.
    If I do not play **Lemmings**, then I’ll study.
    But I failed mathematics.
    Therefore I played **Lemmings**.

19. What conclusion can be drawn from the truth of $\neg p \Rightarrow p$?

20. *For each of the following expressions, use identities to find equivalent expressions which use only* $\wedge$ *and* $\neg$ *and are as simple as possible.*
    (a) $p \vee q \vee \neg r$
    (b) $p \vee (\neg q \wedge r \Rightarrow p)$
    (c) $p \Rightarrow (q \Rightarrow r)$

21. *For each of the following expressions, use identities to find equivalent expressions which use only* $\vee$ *and* $\neg$ *and are as simple as possible.*
    (a) $p \wedge q \wedge \neg r$
    (b) $(p \Rightarrow q \vee \neg r) \wedge \neg p \wedge q$
    (c) $\neg p \wedge \neg q \wedge (\neg r \Rightarrow p)$

22. *Establish the following tautologies by simplifying the left-hand side to the form of the right-hand side:*
    (a) $(p \wedge q \Rightarrow p) \Leftrightarrow$ *true*
    (b) $\neg(\neg(p \vee q) \Rightarrow \neg p) \Leftrightarrow$ *false*
    (c) $((q \Rightarrow p) \wedge (\neg p \Rightarrow q) \wedge (q \Rightarrow q)) \Leftrightarrow p$
    (d) $((p \Rightarrow \neg p) \wedge (\neg p \Rightarrow p)) \Leftrightarrow$ *false*

23. (a) *The* nand *operator (also known to logicians as the* ***Sheffer stroke****), is defined by the following truth table:*

p	q	p nand q
t	t	f
t	f	t
f	t	t
f	f	t

Of course, nand *is a contraction of* ***not-and****;* p nand q *is logically equivalent to* $\neg(p \wedge q)$. *Show that*

i. $(p$ nand $p) \Leftrightarrow \neg p$
ii. $((p$ nand $p)$ nand $(q$ nand $q)) \Leftrightarrow p \vee q$
iii. $((p$ nand $q)$ nand $(p$ nand $q)) \Leftrightarrow p \wedge q$

(b) *Find equivalent expressions for the following, using no connectives other than* nand*:*

i. $p \Rightarrow q$

ii. $p \Leftrightarrow q$

(c) *The* nor *operator (also known to logicians as the* ***Pierce arrow****), is defined by the following truth table:*

p	q	p nor q
t	t	f
t	f	f
f	t	f
f	f	t

For each of the following, find equivalent expressions which use only the nor *operator.*

i. $\neg p$

ii. $p \vee q$

iii. $p \wedge q$

24. *Formalise the second and third proofs of the theorem:*

$$\textit{if} \quad x^2 - 3x + 2 < 0, \ \textit{then} \quad x > 0$$

25. *Write a program to construct truth tables of propositions.*