Predicate Calculus

Jim Woodcock
University of York
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Universal properties

- *language of propositions allows us to model a great variety of properties about specific objects*
- *doesn't allow us to state general properties, such as*
  - *"Every cloud has a silver lining"*
- *these general properties are known as* universal *properties*
- *they describe properties that must be satisfied by* every *individual in some universe of discourse*
- *in our little example*
  - *universe: the set of all clouds*
  - *every one of them has a silver lining*

Examples: universal properties

The following are all examples of universal properties:

1. ***Every** cloud has a silver lining.*
2. ***All** the bells in heaven shall ring.*
3. ***Each** student must hand in homework.*
4. ***Nobody** knows the trouble I seen.*
5. ***Roses** are red.*
6. *Jim doesn't know **anybody** who can sign his bail application.*

Existential properties

- *sometimes we want to express the fact that at least one thing has a particular property*
- *without necessarily knowing which thing*
- *this is known as an* existential *property*

Example

the following are all examples of existential properties:

1. ***something**'s got into the tank*
2. ***there is** a tavern in the town*
3. *i heard it from **one** of your friends*
4. ***a** mad dog has bitten gerald*
5. ***some** people prefer logic*

Symbols

- *two new symbols*
- *"$\forall$" is the universal quantifier*
- *read "for all"*
- *"$\exists$" is the existential quantifier*
- *read "there exists"*
- *quantifier symbols are used to build **predicates***
- *a predicate is like a proposition with various "slots" to be filled in by objects of various kinds*

Example

- *existentially quantified predicate:* "$\exists x : \mathbb{N} \bullet x > 5$"
- *"there is an x, such that $x > 5$"*
- "$x > 5$" *is not a proposition*
- *we can't tell its truth value without knowing x, a "slot"*
- *when we supply a value for x we get a proposition*
- "$0 > 5$" *is clearly false*
- "$\exists x : \mathbb{N} \bullet x > 5$" *is true*
- *since 6 will do for x*

Example: programming languages

- *predicates are propositions with free variables in them*
- *they occur commonly in programming languages*

  if $x > 3$ ***then*** $y := z$

  includes the predicate "$x > 3$".
- *when the statement is executed, the truth value of the assertion "$x > 3$" is determined using the current value of the variable x*
- *the assertion is assigned either the value 1 (representing true) or 0 (representing false)*

Examples

1. Let $Cloud$ be the set of all clouds, and let $SilverLining(x)$ mean that x has a silver lining.

   $$\forall\, c : Cloud \bullet SilverLining(c)$$

2. Let $HolyBell$ be the set of all bells in heaven, and let $Ring(x)$ mean that x shall ring.

   $$\forall\, b : HolyBell \bullet Ring(b)$$

3. Let $Student$ stand for the set of all students, and $Submit(x)$ mean that x must hand in homework.

   $$\forall\, s : Student \bullet Submit(s)$$

4. Let $Person$ be the set of all people, and let $K(x)$ mean that x knows the trouble I seen.

   $$\forall\, p : Person \bullet \neg K(p)$$

5. Let $Rose$ be the set of all roses, and $Red(x)$ mean that x is red.

   $$\forall\, r : Rose \bullet Red(r)$$

6. Let $Person$ be the set of all people, and let $x\ Knows\ y$ mean that x knows y, and $x\ CanVouchFor\ y$ mean that x can sign y's bail application.

   $$\forall\, p : Person \bullet Jim\ Knows\ p \Rightarrow \neg(p\ CanVouchFor\ Jim)$$

   This may also be written as

   $$\forall\, p : Person \mid Jim\ Knows\ p \bullet \neg(p\ CanVouchFor\ Jim)$$

Examples

1. Let $Thing$ stand for the set of all things, and $InTank(x)$ mean that x is in the tank.

   $$\exists\, x : Thing \bullet InTank(x)$$

2. Let $Tavern$ stand for the set of all taverns, and $InTown(x)$ mean that x is in the town.

   $$\exists\, t : Tavern \bullet InTown(t)$$

3. Let $Friends$ stand for the set of all your friends, and $x\ Told\ y$ mean that x has told y.

   $$\exists\, f : Friends \bullet f\ Told\ me$$

4. Let $MadDog$ stand for the set of all mad dogs, and $x\ Bit\ y$ mean that x has bitten y.

   $$\exists\, fido : MadDog \bullet fido\ Bit\ Gerald$$

5. Let $Person$ stand for the set of all people, and $PL(x)$ mean that x prefers logic.

   $$\exists\, p : Person \bullet PL(p)$$

Existential quantification as disjunction

existential quantification may be thought of as a generalised form of disjunction:

$$\exists x : \mathbb{N} \bullet x > 5$$
$$\Leftrightarrow$$
$$(0 > 5) \lor (1 > 5) \lor (2 > 5) \lor (3 > 5) \lor \ldots$$

the predicate must be true for some *natural number*

Universal quantification as conjunction

universal quantification may be thought of as a generalised form of conjunction

$$\forall x : \mathbb{N} \bullet x > 5$$

this is the same as

$$\forall y : \mathbb{N} \bullet y > 5$$

$$\forall x : \mathbb{N} \bullet x > 5$$
$$\Leftrightarrow$$
$$(0 > 5) \land (1 > 5) \land (2 > 5) \land (3 > 5) \land \ldots$$

it must be true for every *natural number.*

Syntax

quantifiers share a similar syntax

$$\clubsuit x : s \mid p \bullet q$$

- $\clubsuit$ *is the* quantifier*;*
- x *is the* bound variable*;*
- s *is the* range *of* x*;*
- p *is the (optional) constraint; and*
- q *is the predicate.*

two syntactic equivalences explain the constraint:

$\exists x : s \mid p \bullet q$ *is a shorthand for* $\exists x : s \bullet p \land q$

$\forall x : s \mid p \bullet q$ *is a shorthand for* $\forall x : s \bullet p \Rightarrow q$

More syntax

the existentially quantified predicate

$$\exists x : s \mid p \bullet q$$

is pronounced "there exists an x *in* s *satisfying* p*, such that* q*"*

the universally quantified predicate

$$\forall x : s \mid p \bullet q$$

is pronounced "for all x *in* s *satisfying* p*,* q *holds"*

Bound variables

- *each quantifier introduces a "bound variable"*
- *like local variable in block-structured programming language*
- *in "$\forall x : s \mid p \bullet q$", the bound variable has a scope that is exactly the constraint p and predicate q*

$$\forall x : s \mid \underbrace{p \bullet q}_{\text{scope of } x}$$

Binding power

- *quantifiers bind very loosely, so*

$$\forall x : s \bullet p \land q$$

  is implicitly parenthesised as "$\forall x : s \bullet (p \land q)$", and not as "$(\forall x : s \bullet p) \land q$".
- *predicates such as "$x > 3$" contain a variable which is yet to be bound by a quantifier: namely, "x"*
- *such yet-to-be-bound variables are called* free variables

Example

the following predicates are formed by universal quantification:

- $\forall x : \mathbb{N} \bullet x < x + 1$
- *"every natural number x is less than $x + 1$"*
- $\forall x : \mathbb{N} \bullet x = 3$
- *"every natural number x is equal to 3"*
- *the first predicate is true, but the second isn't*

Example

- *suppose A is an integer array with 50 entries*
- $A_1, A_2, \ldots, A_{50}$
- *assert that all entries are nonzero*

$$\forall i : \mathbb{N} \bullet 1 \le i \le 50 \Rightarrow A_i \ne 0$$

- *assert entries are sorted in nondecreasing order*

$$\forall i : \mathbb{N} \bullet 1 \le i < 50 \Rightarrow A_i \le A_{i+1}$$

Example

for every number x and every number y, $x + y$ is greater than or equal to x

$$\forall x : \mathbb{N} \bullet \forall y : \mathbb{N} \bullet x + y \geq x$$

Example

x is existentially quantified in the following predicates

- *"there is a number x which is less than $x + 1$"*
- $\exists x : \mathbb{N} \bullet x < x + 1$
- *"there is a number x which is equal to 3"*
- $\exists x : \mathbb{N} \bullet x = 3$
- *"there is a number x which is equal to $x + 1$"*
- $\exists x : \mathbb{N} \bullet x = x + 1$

truth-tables are useless for quantifiers

bound variables may range over sets that are simply too large

Substitution

- *suppose p is a predicate containing free variable x*
- *"$\forall x : S \bullet p$": asserts p is true for every x*
- *specific theorem obtained by* substituting *term t for x*
- *substitution is precise*
- *we write $p[t/x]$ to denote p with term t substituted for x*
- *read as "p with t for x"*
- *substitution also defined on terms themselves: "$u[t/x]$"*

Substitution

- *bound variable must sometimes be renamed before subst*
- $\forall x : \mathbb{Z} \bullet \exists y : \mathbb{Z} \bullet x \neq y$
- *this is true for all numbers, so why not specialise it?*
- *true for y?*

$$\exists y : \mathbb{Z} \bullet y \neq y$$

- *nonsense! — what's gone wrong?*
- *free variable* capture
- free *variable y enters scope of* $\exists y : \mathbb{Z} \bullet \ldots$
- *becomes a* bound *variable*

Validity and Satisfiability

- *any predicate has n arguments*
- *a 0-place predicate is a proposition*
- *if p is an n-place predicate, and values $c_1, \ldots, c_n$ are assigned to each of the individual variables, result is a proposition*
- *p is **valid** if it's true for every choice of arguments*
- *p is **satisfiable** if it's true in some, but not necessarily all, choices of arguments*
- *p is **satisfied** by values $c_1, \ldots, c_n$ that make it true*
- *compare valid, satisfiable, and unsatisfiable predicates with tautologies, contingencies, and contradictions*

Examples: validity, satisfiability, and unsatisfiability

1. *the following predicate on x is **valid**:*

   $$\exists y : \mathbb{N} \bullet y > x$$

   for any natural number x, there exists a larger one
2. *the following predicate is **satisfiable** but not valid:*

   $$\exists y : \mathbb{N} \bullet y < x$$

   for some natural numbers x there exist smaller ones, but not for the number 0
3. *the following predicate is **unsatisfiable***

   $$\exists y : \mathbb{N} \bullet y < x \land y > x$$

   for no natural number can we find a second one that is both larger and smaller than it

Properties of Quantifiers

- *are these two predicates equivalent?*

  $$\exists x : S \bullet p \Rightarrow q \qquad (\exists x : S \bullet p) \Rightarrow (\exists x : S \bullet q)$$

- *does one imply the other?*
- *are they at all related?*

Reasoning with universal quantification I

- *suppose $p(x)$ is true for every possible value of x in s*

  $$\forall x : s \bullet p(x)$$

- *otherwise, $\forall x : s \bullet p(x)$ is false*
- *this gives us an argument: **generalisation***

  $$x \in s, p(x) \vdash \forall x : s \bullet p(x)$$

- ***argument is valid only if premises are true for every x***
- ***x must be arbitrary***
- ***valid:*** $x \in Integer, x + 1 > x \vdash \forall x : Integer \bullet x + 1 > x$
- ***fallacy:*** $x \in Integer, x = 5 \vdash \forall x : Integer \bullet x = 5$

Reasoning with universal quantification II

- *if* $\forall x : s \bullet p(x)$ *is true, then we can choose any term* t *in* s, *and* $p(t)$ *will be true*
- *this gives us an argument:* ***specialisation***

$$\forall x : s \bullet p(x), t \in s \vdash p(t)$$

- $\forall x \in Integer \bullet x^2 \geq 0, y - 1 \in Integer \vdash (y - 1)^2 \geq 0$

Reasoning with existential quantification I

- *suppose* $p(x)$ *is true for at least one element of* s

$$\exists x : s \bullet p(x)$$

- *otherwise,* $\exists x : s \bullet p(x)$ *is false*
- *this gives us an argument*

$$t \in s, p(t) \vdash \exists x : s \bullet p(x)$$

- ***existential introduction***

De Morgan's laws for quantifiers

can this predicate be simplified at all?

$\neg \forall x : \mathbb{N} \bullet x > 3$
$\Leftrightarrow$ *universal quantification as conjunction*
$\neg(0 > 3 \wedge 1 > 3 \wedge 2 > 3 \wedge 3 > 3 \wedge \ldots)$
$\Leftrightarrow$ *by De Morgan's Law*
$\neg(0 > 3) \vee \neg(1 > 3) \vee \neg(2 > 3) \vee \neg(3 > 3) \vee \ldots$
$\Leftrightarrow$ *existential quantification as disjunction*
$\exists x : \mathbb{N} \bullet \neg(x > 3)$

generalised form of De Morgan's Laws for generalised conjunction and disjunction operators

$$(\neg \forall x : S \bullet p) \Leftrightarrow (\exists x : S \bullet \neg p)$$
$$(\neg \exists x : S \bullet p) \Leftrightarrow (\forall x : S \bullet \neg p)$$

Example: reasoning with predicates

- *suppose that* $Person$ *is the set of all people*
- *suppose that if you don't love someone, then you hate them*
- *is it true that there is no one who loves everybody?*
- *we can draw some conclusions from supposing it to be true*
- *first, formalise it as*

$$\neg \exists x : Person \bullet \forall y : Person \bullet x \; loves \; y$$

Let's calculate

$\neg \exists x : Person \bullet \forall y : Person \bullet x \; loves \; y$

$\Leftrightarrow$ *by De Morgan*

$\forall x : Person \bullet \neg \forall y : Person \bullet x \; loves \; y$

$\Leftrightarrow$ *by De Morgan*

$\forall x : Person \bullet \exists y : Person \bullet \neg (x \; loves \; y)$

$\Leftrightarrow$ *by definition*

$\forall x : Person \bullet \exists y : Person \bullet x \; hates \; y.$

so, our sentence means the same as "everybody hates someone"

Example: De Morgan for Quantifiers

- *for every pair of integers x and y, there exists a z such that $x + z = y$*
- *this can be formalised as*

  $\forall x : \mathbb{Z} \bullet \forall y : \mathbb{Z} \bullet \exists z : \mathbb{Z} \bullet x + z = y$

- *or more simply as*

  $\forall x : \mathbb{Z};\; y : \mathbb{Z} \bullet \exists z : \mathbb{Z} \bullet x + z = y$

- *or even more simply as*

  $\forall x, y : \mathbb{Z} \bullet \exists z : \mathbb{Z} \bullet x + z = y$

- *where $\mathbb{Z}$ is the set of Integers*
- *predicate is not true for the positive integers $\mathbb{N}$*

Let's calculate

$\neg \forall x, y : \mathbb{N} \bullet \exists z : \mathbb{N} \bullet x + z = y$

$\Leftrightarrow$ *by De Morgan*

$\exists x, y : \mathbb{N} \bullet \neg \exists z : \mathbb{N} \bullet x + z = y$

$\Leftrightarrow$ *by De Morgan*

$\exists x, y : \mathbb{N} \bullet \forall z : \mathbb{N} \bullet \neg (x + z = y)$

$\Leftrightarrow$ *by definition*

$\exists x, y : \mathbb{N} \bullet \forall z : \mathbb{N} \bullet x + z \neq y$

we can prove it more easily in this form

Finishing it off

- *how do we prove this?*

  $\exists x, y : \mathbb{N} \bullet \forall z : \mathbb{N} \bullet x + z \neq y$

- ***use existential introduction!***
- *find values for x and y that make the following true*

  $\forall z : \mathbb{N} \bullet x + z \neq y$

- *if we select 1 for x and 0 for y, we have*

  $\forall z : \mathbb{N} \bullet 1 + z \neq 0$

  which is a property of every natural number

- *so, given this property, we've proved that*

$$\neg \forall x, y : \mathbb{N} \bullet \exists z : \mathbb{N} \bullet x + z = y$$

Proving and disproving universal quantifications

- ***proving*** *"$\forall x : S \bullet p$" can be quite demanding*
- *we need an argument that proves p **whatever** the value of x*
- *not enough to give some examples of values of x that satisfy p*
- ***disproving*** *"$\forall x : S \bullet p$" is much easier*
- *it's the same as proving "$\exists x : S \bullet \neg p$"*
- *we need only a single y for which p is false*
- *y provides a **counterexample** to "$\forall x : S \bullet p$"*

Example: disproving a universal quantification

- *show the following statement is false*

$$\forall x : \mathbb{Z} \mid x > 0 \bullet x^2 - 3 * x + 2 \geq 0$$

- *a single counterexample will suffice*
- *an appropriate value is $\frac{3}{2}$*
- *we have that*

$$(\tfrac{3}{2})^2 - 3 * (\tfrac{3}{2}) + 2 \quad = \quad -\tfrac{1}{4} \quad < \quad 0$$

Moving quantifier scopes

- *if a predicate occurs in a disjunction or conjunction within the scope of a quantifier, and none of its variables are bound by the quantifier, then it can be removed from the scope of the quantifier*
- *let N stand for a predicate in which x doesn't occur free*
- *we have the following equivalences*

$$(\forall x : S \bullet p \land N) \Leftrightarrow ((\forall x : S \bullet p) \land N)$$
$$(\forall x : S \bullet p \lor N) \Leftrightarrow ((\forall x : S \bullet p) \lor N)$$
$$(\exists x : S \bullet p \land N) \Leftrightarrow ((\exists x : S \bullet p) \land N)$$
$$(\exists x : S \bullet p \lor N) \Leftrightarrow ((\exists x : S \bullet p) \lor N)$$

Moving quantifiers around

- *now suppose that the variable bound by a quantifier occurs in both predicates of a disjunction or conjunction*
- *there are laws for the quantifiers that are analogous to the associativity laws for disjunction and conjunction*
- ***the distributivity laws for the quantifiers***

$$(\forall x : S \bullet p \wedge q) \Leftrightarrow ((\forall x : S \bullet p) \wedge (\forall x : S \bullet q))$$
$$(\exists x : S \bullet p \vee q) \Leftrightarrow ((\exists x : S \bullet p) \vee (\exists x : S \bullet q))$$

Example: distributivity

- *universal quantification doesn't distribute over disjunction*

$$\neg ((\forall x : S \bullet p \vee q) \Leftrightarrow ((\forall x : S \bullet p) \vee (\forall x : S \bullet q)))$$

- ***counterexample***
- *every number is either even or odd*
- *but it's not true that either every number is even, or every number is odd*

Example: distributivity

- *existential quantification doesn't distribute through conjunction*

$$\neg ((\exists x : S \bullet p \wedge q) \Leftrightarrow ((\exists x : S \bullet p) \wedge (\exists x : S \bullet q)))$$

- ***counterexample***
- *there's a number that's even and there's a number that's odd*
- *but there isn't a number that's both even and odd*

Semi-distribution

- *although universal quantification doesn't distribute through disjunction to yield an equivalent predicate, it does yield a **stronger** one*

$$(\forall x : S \bullet p) \vee (\forall x : S \bullet q) \Rightarrow (\forall x : S \bullet p \vee q)$$

- *although existential quantification doesn't distribute through conjunction to yield an equivalent predicate, it does yield a **weaker** one*

$$(\exists x : S \bullet p \wedge q) \Rightarrow (\exists x : S \bullet p) \wedge (\exists x : S \bullet q)$$

Identities and semi-identities

1. $(\forall x : S \bullet p(x)) \Rightarrow p(c)$
2. $p(c) \Rightarrow (\exists x : S \bullet p(x))$
3. $(\forall x : S \bullet \neg p) \Leftrightarrow (\neg \exists x : S \bullet p)$
4. $(\forall x : S \bullet p) \Rightarrow (\exists x : S \bullet p)$
5. $(\exists x : S \bullet \neg p) \Leftrightarrow (\neg \forall x : S \bullet p)$
6. $(\forall x : S \bullet p \wedge N) \Leftrightarrow (\forall x : S \bullet p) \wedge N$
7. $(\forall x : S \bullet p \vee N) \Leftrightarrow (\forall x : S \bullet p) \vee N$

8. $(\forall x : S \bullet p) \wedge (\forall x : S \bullet q) \Leftrightarrow (\forall x : S \bullet p \wedge q)$
9. $(\forall x : S \bullet p) \vee (\forall x : S \bullet q) \Rightarrow (\forall x : S \bullet p \vee q)$
10. $(\exists x : S \bullet p \wedge N) \Leftrightarrow (\exists x : S \bullet p) \wedge N$
11. $(\exists x : S \bullet p \vee N) \Leftrightarrow (\exists x : S \bullet p) \vee N$
12. $(\exists x : S \bullet p \wedge q) \Rightarrow (\exists x : S \bullet p) \wedge (\exists x : S \bullet q)$
13. $(\exists x : S \bullet p) \vee (\exists x : S \bullet q) \Leftrightarrow (\exists x : S \bullet p \vee q)$

Example: multiple quantifiers

- *the order of universal and existential quantifiers is significant*
- *"no matter what value x in S is chosen, a value y in T can be found such that..."*

$$\forall x : S \bullet \exists y : T \bullet \ldots$$

- *value of y may depend on value of x*
- *"no matter what number we choose, there's a larger one"*

$$\forall x : \mathbb{N} \bullet \exists y : \mathbb{N} \bullet y > x$$

Now change the order of the quantifiers

- *"a value y in T can be chosen so that no matter what value x in S is chosen,..."*

$$\exists y : T \bullet \forall x : S \bullet \ldots$$

- *since y is bound first, the value of y must be specified independently of the value of x*
- *"there's a number that's greater than every other number"*

$$\exists y : \mathbb{N} \bullet \forall x : \mathbb{N} \bullet y > x$$

- *which is true and which is false?*

Predicate Calculus Problems

Jim Woodcock
University of York
October 2006

1. *Symbolise the following:*
   - (a) *"Snakes are reptiles."*
   - (b) *"Snakes are not all poisonous."*
   - (c) *"Children are present."*
   - (d) *"Executives all have secretaries."*
   - (e) *"Only executives have secretaries."*
   - (f) *"Only community charge payers may vote in local elections."*
   - (g) *"Employees may use only the goods lift."*
   - (h) *"Only employees may use the goods lift."*
   - (i) *"All that glisters is not gold."*

   - (j) *"All estate agents are not the same."*
   - (k) *"Not all estate agents are the same."*
   - (l) *"None but the brave deserve the fair."*
   - (m) *"Not every visitor stayed for dinner."*
   - (n) *"Not any visitor stayed for dinner."*
   - (o) *"Nothing in the house escaped the children."*
   - (p) *"Some students are both intelligent and hardworking."*
   - (q) *"No coat is waterproof unless it has been specially treated."*
   - (r) *"Some medicines are dangerous only if taken in excessive amounts."*
   - (s) *"All fruits and vegetables are wholesome and nourishing."*
   - (t) *"Everything enjoyable is either immoral, illegal, or fattening."*
   - (u) *"A lecturer is a good teacher if, and only if, he is both well-informed and entertaining."*
   - (v) *"Only university lecturers and firemen are both vastly underpaid and indispensible."*
   - (w) *"Not every actor who is famous is talented."*
   - (x) *"It simply isn't true that every watch will keep good time if and only if it is wound regularly and not abused."*
   - (y) *"Not every person who talks a great deal has a great deal to say."*
   - (z) *"No car that is over ten years old will be mended if it is severely damaged."*

2. *Symbolise the following predicates about the nature of elephants.*
   - (a) *"Any elephant is attractive, if it is neat and well-groomed."*
   - (b) *"Some elephants are gentle and have been well trained."*
   - (c) *"Some elephants are gentle only if they have been well groomed by every student."*
   - (d) *"Some elephants called Jumbo are gentle if they have been well trained."*
   - (e) *"Any elephant is gentle that has been well trained."*
   - (f) *"Any elephant called Jumbo that is gentle has been well trained."*
   - (g) *"No elephant is gentle unless it has been well trained."*
   - (h) *"Any elephant is gentle if it has been well trained."*
   - (i) *"Any elephant has been well trained if it is gentle."*
   - (j) *"Any elephant is gentle if and only if it has been well trained."*
   - (k) *"Gentle elephants have all been well trained."*
   - (l) *"All elephants are called either Jumbo or Dumbo."*
   - (m) *"Every student must ride to graduation on an elephant."*

3. *Symbolise the following predicates about the nature of time.*
   - (a) *"Every instant of time follows another."*
   - (b) *"If two instants of time are not identical, then one follows the other."*
   - (c) *"Time has no beginning."*
   - (d) *"Time has no end."*
   - (e) *"One instant is after a second instant only if the second is before the first.'*

4. *Symbolise the following predicates about the numbers.*
   - (a) *"There's a number between* 3 *and* 5*."*
   - (b) *"Given any number there's a smaller one."*
   - (c) *"There's no biggest number."*
   - (d) *"Addition is commutative."*
   - (e) *"There are two numbers which are such that their product is less than their sum."*
   - (f) *"No cube can be expressed as the sum of two other cubes (unless at least one of the three numbers is zero)."*
   - (g) *"If* $n > 2$*, the equation* $x^n + y^n = z^n$ *cannot be solved in integers* x, y, z*, with* x, y, z *all non-zero."*

5. *Identify the free and bound variables in each of the following.*
   - (a) $(\forall\, x : T \bullet A(x)) \Rightarrow (\exists\, y : U \bullet B(x, y))$
   - (b) $A(x, y) \wedge (\exists\, x : T \bullet B(y)) \Rightarrow (\forall\, y : U;\ z : V \bullet C(x, y, z))$
   - (c) $(\forall\, x : T \bullet \exists\, y : U \bullet A(y, x) \wedge (\forall\, y : V \bullet C(y))) \Rightarrow B(x, y)$
   - (d) $\forall\, x : T;\ y : U \bullet A(z) \Rightarrow B(z)$
   - (e) $A(x) \Rightarrow (B(y) \Rightarrow (\exists\, x : T \bullet C(y) \Rightarrow (\forall\, y : U \bullet D(x))))$

6. *In each of the following, perform the intended substitutions in the corresponding predicates in the last exercise, if the substitutions are legal.*
   - (a) *Substitute* $f(x, z)$ *for* x.
   - (b) *Substitute* z *for* x*, and* $g(y, z)$ *for* y.
   - (c) *Substitute* y *for* x*, and* $f(x, y)$ *for* y.
   - (d) *Substitute* x *for* z.
   - (e) *Substitute* $f(y)$ *for* x*, and* $f(y)$ *for* y.

7. *Express the following as faithfully as possible, using a predicate that starts with a universal or existential quantifier.*
   - (a) *Every noise appals me.*
   - (b) *Something wicked this way comes.*
   - (c) *I have a strange affirmity.*
   - (d) *Their candles are all out.*
   - (e) *He has no children.*
   - (f) *Murders have been performed.*
   - (g) x *is a tale told by an idiot.*
   - (h) *None of woman born shall harm Macbeth.*

8. *Formalise the following propositions. For example, "everyone is married" would be formalised as*

$$\forall x : Person \bullet \exists y : Person \bullet married(x, y)$$

   - (a) *There is someone who is married to everyone else.*
   - (b) *For every integer* x *there is an integer* y *such that the sum of* x *and* y *is* 0.
   - (c) *There is a number* y, *such that for every number* x, *the sum of* x *and* y *is* 0.
   - (d) *No* x *is less than* 0.

9. *Find a predicate* p *in which* x *occurs, and for which* $\forall x : \mathbb{N} \bullet p$ *and* $\exists x : \mathbb{N} \bullet p$ *are both false.*

10. *Find a predicate* p *in which* x *occurs, and for which* $\forall x : \mathbb{N} \bullet p$ *and* $\exists x : \mathbb{N} \bullet p$ *are both true.*

11. *Let the following be defined:*

$N(x)$ *"x is nonnegative"*
$E(x)$ *"x is even"*
$O(x)$ *"x is odd"*
$P(x)$ *"x is prime"*

Formalise the following:
   - (a) *There is an even integer.*
   - (b) *Every integer is either even or odd.*
   - (c) *All prime integers are nonnegative.*
   - (d) *The only even prime is two.*
   - (e) *There is one and only one even prime.*
   - (f) *Not all integers are odd.*
   - (g) *Not all primes are odd.*
   - (h) *If an integer isn't even, then it's odd.*

12. *Give a counterexample to the assertion*

$$(\exists x : \mathbb{N} \bullet p) \wedge (\exists x : \mathbb{N} \bullet q) \vdash \exists x : \mathbb{N} \bullet p \wedge q$$

13. *Give a counterexample to the assertion*

$$\forall x : \mathbb{N} \bullet p \vee q \vdash (\forall x : \mathbb{N} \bullet p) \vee (\forall x : \mathbb{N} \bullet q)$$

14. *Let* A *be a two-dimensional integer array with 20 rows (indexed from 1 to 20), and 30 columns (indexed from 1 to 30). Using the predicate calculus, make the following assertions:*
   - (a) *All entries of* A *are nonnegative.*
   - (b) *All entries of the 4th and 15th rows are positive.*
   - (c) *Some entries of* A *are zero.*
   - (d) *The entries of* A *are sorted into row-major order; that is, the entries are in order within rows, and every entry of the* i*th row is less than or equal to every entry of the* $(i+1)$*th row.*

Predicate Calculus

15. There is another quantifier

$$\exists_1 x : s \bullet p$$

which means "there is a unique x *in* s*, such that* p *holds".*

(a) Define this quantifier in terms of universal and existential quantification.

(b) Universal quantification is a generalisation of conjunction; existential quantification is a generalisation of disjunction. Of what combinator of propositions is the unique quantifier a generalisation?

16. Prove the following laws about quantifiers:

(a) $(\exists x : S \bullet p \Rightarrow q) \Leftrightarrow (\forall x : S \bullet p) \Rightarrow (\exists x : S \bullet q)$

(b) $(\forall x : S \bullet p \Rightarrow q) \Rightarrow ((\forall x : S \bullet p) \Rightarrow (\forall x : S \bullet q))$

(c) $((\exists x : S \bullet p) \Rightarrow (\exists x : S \bullet q)) \Rightarrow (\exists x : S \bullet p \Rightarrow q)$

(d) $(\forall x : S \bullet p \Rightarrow N) \Leftrightarrow (\exists x : S \bullet p) \Rightarrow N$

(e) $(\exists x : S \bullet N \Rightarrow p) \Leftrightarrow N \Rightarrow (\exists x : S \bullet p)$

(f) $(\exists x : S \bullet p \Rightarrow N) \Leftrightarrow (\forall x : S \bullet p) \Rightarrow N$

Predicate Calculus Solutions

Jim Woodcock
University of York
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1. *Symbolise the following:*
   - *(a)* $\forall s : Snake \bullet Reptile(s)$
   - *(b)* $\exists s : Snake \bullet \neg Poisonous(s)$
   - *(c)* $\exists c : Person \mid Child(c) \bullet Present(c)$
   - *(d)* $\forall e : Person \mid Executive(e) \bullet HasSecretary(e)$
   - *(e)* $\forall e : Person \mid HasSecretary(e) \bullet Executive(e)$
   - *(f)* $\forall p : Person \mid Vote(p) \bullet PollTax(p)$
   - *(g)* $\forall p : Person; \ l : Lift \mid Employee(p) \land p \ UsesLift \ l \bullet Goods(l)$
   - *(h)* $\forall p : Person; \ l : Lift \mid p \ UsesLiift \ l \land Goods(l) \bullet Employee(p)$
   - *(i)* $\neg \forall t : Thing \mid Glisters(t) \bullet Gold(t)$

   - *(j)* $\forall p, q : Person \mid EstateAgent(p) \land EstateAgent(q) \bullet p \neq q$
   - *(k)* $\neg \forall p, q : Person \mid EstateAgent(p) \land EstateAgent(q) \bullet p = q$
   - *(l)* $\forall p : Person \mid \neg Brave(p) \bullet \neg DeservesFair(p)$
   - *(m)* $\neg \forall p : Person \mid Visitor(p) \bullet Dined(p)$
   - *(n)* $\neg \exists p : Person \mid Visitor(p) \bullet Dined(p)$
   - *(o)* $\neg \exists t : Thing \mid InHouse(t) \bullet EscapedKids(t)$
   - *(p)* $\exists s : Student \bullet Intelligent(s) \land HardWorking(s)$
   - *(q)* $\neg \exists c : Coat \mid Waterproof(c) \bullet \neg Treated(c)$
   - *(r)* $\exists m : Medicine \bullet Dangerous(m) \Rightarrow TooMuch(m)$
   - *(s)* $\forall x : Food \mid Fruit(x) \lor Veg(x) \bullet Wholesome(x) \land Nourishing(x)$
   - *(t)* $\forall t : Thing \mid Enjoyable(t) \bullet$
     $Immoral(t) \lor Illegal(t) \lor Fattening(t)$
   - *(u)* $\forall l : Person \mid Lecturer(l) \bullet$
     $GoodTeacher(l) \Leftrightarrow WellInformed(l) \land Entertaining(l)$
   - *(v)* $\forall p : Person \mid VastlyUnderpaid(p) \land Indispensible(p) \bullet$
     $Lecturer(p) \lor Fireman(p)$
   - *(w)* $\neg \forall a : Actor \mid Famous(a) \bullet Talented(a)$
   - *(x)* $\neg \forall w : Watch \bullet$
     $GoodTimekeeper(w) \Leftrightarrow RegularlyWound(w) \land \neg Abused(w)$
   - *(y)* $\neg \forall p : Person \mid TalksALot(p) \bullet SaysAGreatDeal(p)$
   - *(z)* $\neg \exists c : Car \mid Age(c) > 10 \land SeverelyDamaged(c) \bullet$
     $ToBeMended(c)$

2. (a) $\forall e : Elephant \mid Neat(e) \land WellGroomed(e) \bullet Attractive(e)$
   (b) $\exists e : Elephant \bullet Gentle(e) \land WellTrained(e)$
   (c) $\exists e : Elephant \bullet Gentle(e) \Rightarrow (\forall s : Student \bullet s\ HasWellGroomed\ e)$
   (d) $\exists e : Elephant \mid Jumbo(e) \bullet WellTrained(e) \Rightarrow gentle(e)$
   (e) $\forall e : Elephant \mid WellTrained(e) \bullet Gentle(e)$
   (f) $\forall e : Elephant \mid Jumbo(e) \land WellTrained(e) \bullet Gentle(e)$
   (g) $\neg \exists e : Elephant \mid Gentle(e) \bullet \neg WellTrained(e)$
   (h) $\forall e : Elephant \mid WellTrained(e) \bullet Gentle(e)$
   (i) $\forall e : Elephant \mid Gentle(e) \bullet WellTrained(e)$
   (j) $\forall e : Elephant \bullet Gentle(e) \Leftrightarrow WellTrained(e)$
   (k) $\forall e : Elephant \mid Gentle(e) \bullet WellTrained(e)$
   (l) $\forall e : Elephant \bullet Jumbo(e) \lor Dumbo(e)$
   (m) $\forall s : Student \bullet (\exists e : Elephant \bullet s\ RidesToGraduationOn\ e)$

3. (a) $\forall t : Time \bullet (\exists u : Time \bullet u\ isbefore\ t)$
   (b) $\forall t, u : Time \mid t \neq u \bullet t\ isbefore\ u \lor u\ isbefore\ t$
   (c) $\neg \exists t : Time \bullet (\forall u : Time \bullet t\ isbefore\ u)$
   (d) $\neg \exists t : Time \bullet (\forall u : Time \bullet u\ isbefore\ t)$
   (e) $\forall t, u : Time \mid u\ isafter\ t \bullet t\ isbefore\ u$

4. (a) $\exists n : \mathbb{Z} \bullet 3 < x < 5$
   (b) $\forall n : \mathbb{Z} \bullet (\exists m : \mathbb{Z} \bullet m < n)$
   (c) $\neg \exists n : \mathbb{Z} \bullet (\forall m : \mathbb{Z} \bullet m \leq n)$
   (d) $\forall x, y : \mathbb{Z} \bullet x + y = y + x$
   (e) $\exists m, n : \mathbb{Z} \bullet m * n < m + n$
   (f) $\neg \exists x, y, z : \mathbb{Z} \mid x \neq 0 \land y \neq 0 \land z \neq 0 \bullet x^3 = y^3 + z^3$
   (g) $\forall n : \mathbb{N} \mid n > 2 \bullet (\neg \exists x, y, z : \mathbb{Z} \mid x \neq 0 \land y \neq 0 \land z \neq 0 \bullet x^n = y^n + z^n)$

5. (a) $(\forall \overset{b}{x} : T \bullet A(\overset{b}{x})) \Rightarrow (\exists \overset{b}{y} : U \bullet B(\overset{f}{x}, \overset{b}{y}))$
   (b) $A(\overset{f}{x}, \overset{f}{y}) \land (\exists \overset{b}{x} : T \bullet B(\overset{f}{y})) \Rightarrow (\forall \overset{b}{y} : U;\ \overset{b}{z} : V \bullet C(\overset{f}{x}, \overset{b}{y}, \overset{b}{z}))$
   (c) $(\forall \overset{b}{x} : T \bullet \exists \overset{b^1}{y} : U \bullet A(\overset{b^1}{y}, \overset{b}{x}) \land (\forall \overset{b^2}{y} : V \bullet C(\overset{b^2}{y}))) \Rightarrow B(\overset{f}{x}, \overset{f}{y})$
   (d) $\forall \overset{b}{x} : T;\ \overset{b}{y} : U \bullet A(\overset{f}{z}) \Rightarrow B(\overset{f}{z})$
   (e) $A(\overset{f}{x}) \Rightarrow (B(\overset{f}{y}) \Rightarrow (\exists \overset{b}{x} : T \bullet C(\overset{f}{y}) \Rightarrow (\forall \overset{b}{y} : U \bullet D(\overset{b}{x}))))$

6. (a) $(\forall x : T \bullet A(x)) \Rightarrow (\exists y : U \bullet B(\underline{f(x,z)}, y))$
   (b) *This substitution is not legal, since there would be variable capture.*
   (c) $(\forall x : T \bullet \exists y : U \bullet A(y, x) \land (\forall y : V \bullet C(y)) \Rightarrow B(\underline{y}, \underline{f(x,y)})$
   (d) *This substitution is not legal, since there would be variable capture.*
   (e) *This substitution is not legal, since there would be variable capture.*

7. (a) $\forall\, n : Noise \bullet AppalsMe(n)$
   (b) $\exists\, t : Thing \mid Wicked(t) \bullet ThisWayComes(t)$
   (c) $\exists\, a : Affirmity \mid Strange(a) \bullet IHave(a)$
   (d) $\forall\, c : TheirCandles \bullet Out(c)$
   (e) $\forall\, p : Person \mid Child(p) \bullet \neg His(p)$
   (f) $\exists\, m : Murder \bullet Performed(m)$
   (g) $\exists\, i : Person \mid Idiot(i) \bullet i\ told\ x$
   (h) $\forall\, p : Person \mid OfWomanBorn(p) \bullet \neg p\ ShallHarm\ Macbeth$

8. (a) $\exists\, p : Person \bullet (\forall\, q : Person \mid q \neq p \bullet married(p, q))$
   (b) $\forall\, x : \mathbb{Z} \bullet (\exists\, y : \mathbb{Z} \bullet x + y = 0)$
   (c) $\exists\, y : \mathbb{Z} \bullet (\forall\, x : \mathbb{Z} \bullet x + y = 0)$
   (d) $\neg \exists\, x : \mathbb{Z} \bullet x < 0$

9. $x \neq x$

10. $x = x$

11. (a) $\exists\, x : \mathbb{Z} \bullet E(x)$
    (b) $\forall\, x : \mathbb{Z} \bullet E(x) \vee O(x)$
    (c) $\forall\, x : \mathbb{N} \mid P(x) \bullet N(x)$
    (d) $\forall\, x : \mathbb{N} \mid E(x) \wedge P(x) \bullet x = 2$
    (e) $\exists\, x : \mathbb{N} \mid E(x) \wedge P(x) \bullet (\forall\, y : \mathbb{N} \mid E(y) \wedge P(y) \bullet y = x)$
    (f) $\neg \forall\, x : \mathbb{Z} \bullet O(x)$
    (g) $\neg \forall\, x : \mathbb{Z} \mid P(x) \bullet O(x)$
    (h) $\forall\, x : \mathbb{Z} \mid \neg E(x) \bullet O(x)$

12. *Take p to be "x is even", and q to be "x is odd".*

13. *take p and q as in the answer to part 12.*

14. (a) $\forall\, i, j : \mathbb{N} \mid 1 \leq i \leq 20 \wedge 1 \leq j \leq 30 \bullet A_{i,j} \geq 0$
    (b) $\forall\, j : \mathbb{N} \mid 1 \leq j \leq 30 \bullet A_{4,j} > 0 \wedge A_{15,j} > 0$
    (c) $\exists\, i, j : \mathbb{N} \mid 1 \leq i \leq 20 \wedge 1 \leq j \leq 30 \bullet A_{i,j} = 0$
    (d) $\forall\, i : \mathbb{N} \mid 1 \leq i \leq 20 \bullet \forall\, j : \mathbb{N} \mid 1 \leq j < 29 \bullet A_{i,j} \leq A_{i,j+1}$
    $\forall\, i : \mathbb{N} \mid 1 \leq i < 19 \bullet \forall\, j : \mathbb{N} \mid 1 \leq j \leq 30 \bullet A_{i,j} \leq A_{i+1,j}$

15. *(a)* $\exists x : s \bullet p(x) \wedge (\forall y : s \bullet p(y) \Rightarrow y = x)$

(b) It is a generalisation of exclusive-or.

16. *(a)*

$\exists x : S \bullet p \Rightarrow q$
$\Leftrightarrow$ *implication*
$\quad \exists x : S \bullet \neg p \vee q$
$\Leftrightarrow$ $\exists$-$\vee$ *distribution*
$\quad (\exists x : S \bullet \neg p) \vee (\exists x : S \bullet q)$
$\Leftrightarrow$ *generalised De Morgan*
$\quad \neg(\forall x : S \bullet p) \vee (\exists x : S \bullet q)$
$\Leftrightarrow$ *distribution*
$(\forall x : S \bullet p) \Rightarrow (\exists x : S \bullet q)$

(b) Let x be an arbitrary member of S, and consider first:

$(\forall x : S \bullet p \Rightarrow q) \wedge (\forall x : S \bullet p)$
$\Rightarrow$ *specialisation*
$\quad (p \Rightarrow q) \wedge (\forall x : S \bullet p)$
$\Rightarrow$ *specialisation*
$\quad (p \Rightarrow q) \wedge p$
$\Rightarrow$ *propositional calculus*
$\quad q$
$\Rightarrow$ *generalisation*
$\quad \forall x : S \bullet q$

We've proved something of the form $(s \wedge t) \Rightarrow u$, which is, by 'exportation', equivalent to $(s \Rightarrow t) \Rightarrow u$. Hence we have shown the required result.

$(\forall x : S \bullet p \Rightarrow q) \Rightarrow$
$\quad ((\forall x : S \bullet p) \Rightarrow (\forall x : S \bullet q))$

(c)

$(\exists x : S \bullet p) \Rightarrow (\exists x : S \bullet q)$
$\Leftrightarrow$ *implication*
$\quad \neg(\exists x : S \bullet p) \vee (\exists x : S \bullet q)$
$\Leftrightarrow$ *generalised De Morgan*
$\quad (\forall x : S \bullet \neg p) \vee (\exists x : S \bullet q)$
$\Rightarrow$ *forall implies exists, with monotonicity*
$\quad (\exists x : S \bullet \neg p) \vee (\exists x : S \bullet q)$
$\Leftrightarrow$ $\exists - \vee$ *distribution*
$\quad \exists x : S \bullet \neg p \vee q$
$\Leftrightarrow$ *implication*
$\quad \exists x : S \bullet p \Rightarrow q$

(d)

$\forall x : S \bullet p \Rightarrow N$
$\Leftrightarrow$ *implication*
$\quad \forall x : S \bullet \neg p \vee N$
$\Leftrightarrow$ *no capture on* $\vee$
$\quad (\forall x : S \bullet \neg p) \vee N$
$\Leftrightarrow$ *generalised De Morgan*
$\quad \neg(\exists x : S \bullet p) \vee N$
$\Leftrightarrow$ *implication*
$\quad (\exists x : S \bullet p) \Rightarrow N$

(e)

$\exists x : S \bullet N \Rightarrow p$
$\Leftrightarrow$ *implication*
$\quad \exists x : S \bullet \neg N \vee p$
$\Leftrightarrow$ *no capture on* $\vee$
$\quad \neg N \vee \exists x : S \bullet p$
$\Leftrightarrow$ *implication*
$\quad N \Rightarrow \exists x : S \bullet p$

(f)

$\exists x : S \bullet p \Rightarrow N$
$\Leftrightarrow$ *implication*
$\quad \exists x : S \bullet \neg p \vee N$
$\Leftrightarrow$ *no capture on* $\vee$
$\quad (\exists x : S \bullet \neg p) \vee N$
$\Leftrightarrow$ *generalised De Morgan*
$\quad \neg(\forall x : S \bullet p) \vee N$
$\Leftrightarrow$ *implication*
$\quad (\forall x : S \bullet p) \Rightarrow N$