The structure of partial isometries I

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Abstract

This paper is an investigation of the structure of partial isometries. We consider the order theory and categorical structure, in terms of the 'quantum logic' and 'compact closed categories' approaches to the foundations of quantum mechanics.

1 Introduction

Already in 1936, von Neumann and Birkhoff proposed treating projectors on Hilbert space as propositions about quantum systems [9]. The set of all projectors on a Hilbert space forms a complemented lattice, and – following a long tradition in order theory – the lattice operations *meet*, *join*, and *complement* were thus interpreted as the logical connectives *conjunction*, *disjunction* and *negation*.

However, the lattice of projectors is not a boolean lattice, so this interpretation as connectives requires modifications to the rules of propositional logic (notably the replacement of the distributive law, $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ with a significantly weaker condition). The resulting system, based on *orthomodular lattices*, has become known as *quantum logic*, and a number of authors [19,54] have suggested that the non-classical behaviour of quantum systems simply results from the fact that orthomodular lattices, rather than boolean lattices provide the natural logic of quantum systems.

This paper does not take a position on this claim — although we do discuss arguments for and against a logical interpretation of orthomodular lattices. Rather we use the lattice structure of projectors in order to study (from both categorical and order-theoretic perspectives) a natural generalisation. Precisely, although measurements (as projectors) are well-studied, we wish

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[1] Research supported by EU FP6 grant : QICS

to consider the theory of combinations of dynamical processes (considered as unitary maps) and measurements (considered as projectors).

To this end, we study *partial isometries.* These generalise both projectors and unitaries in a natural way, and we demonstrate in Corollary 10 that for finite-dimensional spaces, they may be characterised as the composite of a unitary map and a projector (the infinite-dimensional case is, not unexpectedly, more complex). They have a natural partial order introduced by Halmos and McLaughlin in [27], and when restricted to projectors this is exactly the partial order of the orthomodular lattice of 'quantum propositions'

This partial order allows us to study partial isometries from a category-theoretic point of view. It is well-known [18] that the composite of two partial isometries is not, in general, a partial isometry. However, there is a natural associative composition (based on the conjunction of quantum logic, and closely related to the treatment of partial isometries in inverse semigroup theory) that allows us to define a category of partial isometries. The resulting category is shown to be an *inverse category* — inverse categories have a natural partial order on their hom-sets, and we demonstrate that this is exactly the Halmos-McLaughlin partial order.

We then consider the structure of this category, from the point of view of the alternative *categorical foundations* for quantum mechanics of [5], and from the point of view of categorical logic generally. We demonstrate that our category of partial isometries is a compact closed category, with respect to the tensor product, and consider both logical interpretations, and connections to the categorical foundations approach to quantum mechanics.

Finally, we study the domain theory of the hom-sets of this cateory. The Halmos-McLaughlin partial order (i.e. the inverse-semigroup theoretic natural partial order) is demonstrated to give a Scott domain structure to the hom-sets. Moreover, the down-closure of each maximal element is demonstrated to be an orthomodular lattice. This is strongly reminiscent of the order-theoretic appoach to *coherent spaces* [], used to model Linear Logic [21]. In [11], it is demonstrated that coherent spaces may be characterised as special types of Scott domains where, beneath every maximal element, there is a boolean lattice. Hence, the natural (inverse semigroup theoretic) order theory of partial isometries may be considered to be analogous to coherent spaces, where the classical logic (i.e. the boolean lattice) has been replaced by a quantum logic (i.e. the orthomodular lattice).

2 The order theory of projectors, and 'quantum logic'

The order theory of projectors on Hilbert space is the foundation of the 'quantum logic' of Birkhoff / von Neumann [9]. The partial order on projectors is defined as follows :

Definition 1 The lattice of projectors on a Hilbert space :
Let $E, F : H \rightarrow H$ *be projectors on a Hilbert space* H. *We say that* E *is* **below** F, *written* $E \leq F$ *when* $EF = E$. *Note that this implies* $EF = FE$. *It is straightforward that* $\leq$ *is a partial order, and the set of all projectors on* H *forms a lattice, with top element the identity map* 1_H *and bottom element the zero map* 0_H.

The meet and join of this lattice may be given explicitly :

Proposition 2 Partial orders and meets on projectors :
Let E, F *be projectors on some Hilbert space, corresponding to the subspaces* H_E, H_F *respectively. Then*

(1) *The* join $E \vee F$ *is defined by*

$$E \vee F = Inf\{G : E \leq G \text{ and } F \leq G\}$$

and is simply the projector onto the smallest subspace containing both H_E *and* H_F.

(2) *The* meet $E \wedge F$ *is defined by*

$$E \wedge F = Sup\{G : G \leq E \text{ and } G \leq F\}$$

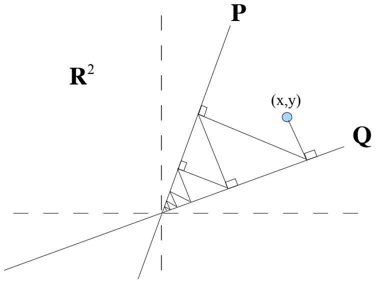
and is the projector onto the largest subspace contained within both H_E *and* H_F. *It may be given given explicitly by* $E \wedge F = lim_{n \rightarrow \infty}(EF)^N = lim_{n \rightarrow \infty}(FE)^N$.

PROOF. We refer to [39] for these results, and [52] for a proposed physical interpretation of the meet operation. We illustrate the process in $\mathbb{R}^2$, with two 1-dimensional subspaces P, Q in Figure 2. $\Box$

All lattices of projectors on Hilbert space are shown in [51] to be *orthomodular lattices*, defined as follows :

Definition 3 orthomodular lattices, orthocomplemented lattices :
Let $L, \leq$ *be a complete lattice. We say that it is* **orthocomplemented** *when there exists an involution* $(\ )^\perp : L \rightarrow L$ *that satisfies*

Fig. 1. The meet of two projectors, in $\mathbb{R}^2$



(1) $A \vee A^{\perp} = \top$.
(2) $A \wedge A^{\perp} = \perp$.
(3) $A \leq B$ *if and only if* $B^{\perp} \leq A^{\perp}$.

An orthocomplemented lattice L *is called* **orthomodular** *if it satisfies the additional condition*

(4) $A \leq B$ *implies* $B = A \vee (A^{\perp} \wedge B)$

The canonical example of an orthomodular lattice is the lattice of projectors on a Hilbert space (in fact, we refer to [**?**] for a stronger statement). The involution of the projector onto a subspace $S \leq H$ is simply the projector onto the orthogonal complement $S^{\perp} = \{h \in H : \langle s|h\rangle = 0 \ , \ \forall \ s \in S\}$.

2.0.0.1 Orthomodular lattices as logics Following a long-established tradition in order theory, 'quantum logic' treats the elements of a lattice of projectors as propositions, and *meet* and *join* as *conjunction* and *disjunction* respectively (although note the failure of the distributive law described in Section 1). The involution is interpreted as negation, and satisfies analogues of de Morgan's laws $(P \wedge Q)^{\perp} \ = \ P^{\perp} \vee Q^{\perp}$.

At this point, readers familiar with mathematical logic will naturally wonder about both *negation* and *implication*. Following order-theoretic approaches to classical logic, a connective (the *Sasaki hook*) may be defined in terms of conjunction, disjunction and negation, as $P \overset{S}{\rightarrow} Q \ = \ P^{\perp} \vee (P \wedge Q)$. However, as discussed in Section **??**, it is controversial whether this should be accepted as a genuine implication. We refer to [30] for more properties, and a strong defence of this connective as a form of implication.

2.0.0.2 Quantum logic and foundations of quantum mechanics We emphasise that we are <u>not</u> taking a stand on 'quantum logic' as a foundation for quantum mechanics. Our approach is simply to study the order-theoretic structure of various aspects of Hilbert space.

For an introduction to quantum mechanics via quantum logic, we refer to [31] for an excellent exposition. We also refer to [32] for the related, but conflicting, 'consistent histories' approach to foundations, and to [58,1] for connections between order theory and logic in the classical world.

3 Partial Isometries

We give the definition of partial isometries, and various simple properties.

3.1 Definitions

The following definitions are taken from [27]

Definition 4 partial isometries, initial and final subspaces, isometries :
Let $L : H_1 \rightarrow H_2$ *be a linear map of Hilbert spaces, and denote its adjoint by* $L^* : H_2 \rightarrow H_1$*. Then* L *is a* **partial isometry** *when* $L^*L : H_1 \rightarrow H_1$ *is a projector, and hence (or equivalently)* $LL^* : H_2 \rightarrow H_2$ *is a projector.*

The projectors $E_L = L^*L : H_1 \rightarrow H_1$ *and* $F_L = LL^* : H_2 \rightarrow H_2$ *are called the* **initial** *and* **final projectors** *of* L*, and the corresponding subspaces are the* **initial** *and* **final subspaces**. *When the initial subspace is the whole of* H_1*, then* L *is called an* **isometry**.

Given a partial isometry L, it is immediate that L^* is also a partial isometry L, and the initial projector of L is the final projector of L^*. Both unitary maps and projectors are, trivially, partial isometries. The initial and final projectors of a unitary map are the global identities on its source and target space, and a projector is its own initial and final projector.

3.2 Basic properties

We establish some basic algebraic results on partial isometries :

Proposition 5 Standard results on partial isometries
Given partial isometries $L : H_1 \rightarrow H_2$, $M : H_2 \rightarrow H_3$, $N : K_1 \rightarrow K_2$, *then :*

(1) L *is a unitary map between its initial and final subspaces.*
(2) $LL^*L = L$ *and* $L^*LL^* = L^*$
(3) When the initial and projectors of $L : H_1 \rightarrow H_2$ *are the global identities of* H_1 *and* H_2 *respectively, then* L *is a unitary map.*

(4) $ML : H_1 \to H_3$ *is a partial isometry exactly when the initial projector of* M *commutes with the final projector of* L*, so* $E_M F_L = F_L E_M$.
(5) $L \oplus K : H_1 \oplus K_1 \to H_2 \oplus K_2$ *is a partial isometry.*
(6) $L \otimes K : H_1 \otimes K_1 \to H_2 \otimes K_2$ *is a partial isometry.*

PROOF. Results (1)-(4) are taken from [18]), and Results (5) and (6) are a simple consequence of linearity. Note that (2) states that the adjoint $(\ )^*$ is a *generalised inverse* (in the sense of semigroup theory [46]) on partial isometries. However, the set of all partial isometries on a Hilbert space H is neither a regular nor an inverse semigroup, since by (4), it is not closed under composition. □

Corollary 6 *The composite of a unitary and a partial isometry is always a partial isometry.*

PROOF. The initial and final projectors of a unitary map $U : H \to K$ are the global identities on H, K respecitively. The result then follows trivially from (4) above. □

3.3 The Halmos-McLaughlin partial order

The partial order on projectors given in Section 2 is a special case of the partial order on partial isometries given in [27] :

Definition 7 The Halmos-McLaughlin partial order
The $\leqq$ *on partial isometries is defined in [27] by*

$$L \leqq K \ \Leftrightarrow \ L = KE_L$$

i.e. K*, when restricted to the initial subspace of* L*, is exactly* L*. It is then immediate* $\leqq$*, when restricted to projectors, is exactly the partial ordering of Definition 1. We refer to* $\leqq$ *as the* **Halmos-McLaughlin partial order***, or* **HML partial order***.*

We may then characterise the projectors on a space H as 'partial isometries beneath the identity 1_H'. From the physical interpretation given in Appendix **??**, we have a particular interest in partial isometries that are beneath unitary maps :

Definition 8 Physical partial isometries
Given a partial isometry $L : H_1 \to H_2$ *satisfying* $L \leqq U$ *for some unitary map* $U : H_1 \to H_2$*, we refer to* L *as a* **physical** *partial isometry.*

Proposition 9 *Let $L : H_1 \to H_2$ be a partial isometry. When the codimension of the initial subspace is equal to the codimension of the final subspace, then L is a physical partial isometry.*

PROOF. Denote the initial subspace of $L : H_1 \to H_2$ by S, and the terminal subspace by T, so $H_1 = S \oplus S^\perp$ and $H_2 = T \oplus T^\perp$. By definition of partial isometries $dim(S) = dim(T)$, and the condition on the codimensions gives that $dim(S^\perp) = dim(T^\perp)$, and so $S^\perp \cong T^\perp$. Given a (not necessarily unique) unitary $L' : S^\perp \to T^\perp$ exhibiting this isomorphism, we may construct a unitary $U = L + \begin{pmatrix} 0 & 0 \\ 0 & L' \end{pmatrix} : S \oplus S^\perp \to T \oplus T^\perp$, and it is immediate from the definition that $L \leqq U$. □

Corollary 10 *All partial isometries on finite-dimensional spaces are physical isometries.*

PROOF. The condition on codimensions from Proposition 9 is trivially satisfied for partial isometries between finite-dimensional spaces. Counterexamples on infinite-dimensional spaces include the Cuntz-Krieger algebras of [13], and the Shift operator of [25]. □

3.4 The interaction of partial isometries and projectors

We now introduce a useful technique for dealing with the interaction of partial isometries and projectors. This is strongly based on a common inverse-semigroup theoretic (see, for example, [28]). However, we emphasise that, with the usual composition of linear maps, partial isometries do not form an inverse semigroup - they are not even closed under composition.

Proposition 11 "Pushing a projector through a partial isometry."
Let $L : H_1 \to H_2$ be a partial isometry, and let $G : H_1 \to H_1$ and $D : H_2 \to H_2$ be projectors satisfying $G \leq E_L$ and $D \leq F_L$. Then

(1) there exists a unique projector $G' \leq F_L$ such that $LG = G'L$.
(2) there exists a unique projector $D' \leq E_L$ such that $DL = LD'$.

PROOF.

(1) Define $G' : H_2 \to H_2$ by $G' = LGL^*$.Then $(G')^* = (LGL^*)^* = LGL^*$, so G' is self-adjoint. Similarly,

$$G'G' = LGL^*LGL^* = LGE_LGL^* = LGGL^* = LGL^* = G'$$

so G' is idempotent, and hence it is a projector. Now note

$$G'L = LGL^*L = LGE = LEG = LL^*LG = LG$$

as required. To show that $G' \leq L_F$, note that

$$G'F = G'LL^* = LGL^*LL^* = LGEL^* = LEL^* = LL^*LL^* = F^2 = F$$

Uniqueness follows since L is a unitary when restricted to its initial / final subspaces.

(2) Defining $D' : H_1 \to H_1$ by $D' = L^*DL$, this result follows by symmetry.

□

This 'pushing a projector through a partial isometry' operation is order-preserving, as shown :

Lemma 12 *Let $L : H_1 \to H_2$ be a partial isometry, and let $P, Q : H_2 \to H_2$ be projectors below F_L. Then the unique projectors $P', Q' \leq E_L$ satisfying*

$$PL = LP' \quad , \quad QL = LQ'$$

satisfy $P \leq Q \Leftrightarrow P' \leq Q'$.

PROOF. ($\Rightarrow$) Assume $P \leq Q$, so $PQ = QP = P$. By construction $P' = L^*PL$ and $Q' = L^*QL$ so

$$P'Q' = L^*PLL^*QL = L^*LL^*PQL = L^*PQL = L^*PL = P'$$

and hence $P' \leq Q'$.
($\Leftarrow$) This proof is almost identical to ($\Rightarrow$) above. □

The technique of 'pushing a projector through a partial isometry allows us to establish a connection between partial isometries and block matrices, as follows :

Proposition 13 *Let $U : H \to K$ be a unitary map, let $H = H_1 \oplus \ldots \oplus H_a$ and $K = K_1 \oplus \ldots K_b$ be direct sum decompositions of the source and target* *space. We may then write U as a $(b \times a)$ block matrix*

$$U = \begin{pmatrix} U_{11} & \dots & U_{1a} \\ \vdots & \ddots & \vdots \\ U_{b1} & \dots & U_{ba} \end{pmatrix} \quad \textit{where } \; U_{ij} : H_j \to K_i \;\; \forall \; 1 \leq i \leq a \; , \; 1 \leq j \leq b$$

For fixed i, j, the submatrix $B_{ij} = \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} \\ \vdots & U_{ij} & \vdots \\ \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}$ is given by $B_{ij} = QUP$ where $Q = \mathbf{0} \oplus 1_{K_i} \oplus \mathbf{0} : K \to K$ and $P = \mathbf{0} \oplus 1_{H_j} \oplus \mathbf{0} : H \to H$. This is then a partial isometry exactly when

$$Q'P = PQ' \quad \textit{or equivalently, } QP' = P'Q$$

where $Q' = U^{-1}QU$ and $P' = UPU^{-1}$ are given by 'passing the projector P (resp. Q) through the unitary U', as in Proposition 11.

PROOF. Given $B_{ij} = QUP : H \to K$, then since U is unitary, $F_U = 1_K$, and $Q \leqq F_U$. Therefore, by Proposition 11, $QU = UQ'$, and this is a partial isometry with initial projector Q'. Hence, by Proposition 5, $QUP = UQ'P$ is a partial isometry exactly when $Q'P = PQ'$, as required. The equivalent condition $QP' = PQ$ follows either algebraically (by conjugation by U), or by duality. □ □

3.4.0.3 Interpretation We may consider a quantum computation that consists of a finite series of operations — either unitary maps, or measurements on certain subspaces. A natural question would be whether this is equivalent to a series of unitaries (and by composition, a single unitary map) followed by a measurement. However, a unitary followed by a projector is a partial isometry, whereas a series of unitaries and projectors is not, in general. Proposition 13 gives conditions for a projector, followed by a unitary, followed by a projector to be a partial isometry (i.e. equivalent to a unitary followed by a projector), and this is easily generalised.

4 A category of partial isometries

From Proposition 5, the class of partial isometries is *not* closed under the usual composition of linear maps. However, with a modified composition (that we demonstrate is equivalent to a construction of [46]), partial isometries are not only closed under composition, but form an *inverse category.*

Definition 14 Inverse Categories, Generalised Inverses
Inverse categories are the natural extension of inverse monoids to the many-object case [46]. A category $\mathbf{C}$ *is* **inverse** *when for every arrow* $f \in \mathbf{C}(X, Y)$*, there exists a unique* **generalised inverse** $f^{-1} \in \mathbf{C}(Y, X)$ *satisfying*

$$ff^{-1}f = f \quad \textit{and} \quad f^{-1}ff^{-1} = f^{-1}$$

We emphasise that this definition does not *require* $f^{-1}f = 1_X$ *or* $ff^{-1} = 1_Y$*. Generalised inverses that satisfy these additional conditions are called* **left** *and* **right global inverses** *respectively.*

In order to define the category of partial isometries, we give a binary operation on partial isometries that we will prove is the composition in a category :

Definition 15 *Given partial isometries* $L : H_1 \to H_2$ *and* $M : H_2 \to H_3$*, we define*

$$M \circ L = lim_{n\to\infty} \left[(ML)(ML)^*\right]^n (ML) = lim_{n\to\infty}(ML) \left[(ML)^*(ML)\right]^n$$

For readers familiar with Girard's Geometry of Interaction system [22,23], this definition is clearly motivated by the *execution formula*[2] . However, it also has a close connection with the conjunction in quantum logic :

Lemma 16 *Given partial isometries* $M : H_2 \to H_3$*,* $L : H_1 \to H_2$*, as above, then* $M \circ L = M(E_M \wedge F_L)L : H_1 \to H_3$ *where* E_M *and* F_L *are the final and initial projectors of* M *and* L *respectively. Hence* $M \circ L$ *exists for arbitrary partial isometries* $L : H_1 \to H_2$ *and* $M : H_2 \to H_3$*, and is a partial isometry.*

Also, when $E_MF_L = F_LE_M$*, then* $M \circ L$ *is simply* ML*, the usual composition of* M *and* L *as linear maps.*

PROOF. From Proposition 2,

$$E_M \wedge F_L = lim_{n\to\infty}(E_MF_L)^n = lim_{n\to\infty}(F_LE_M)^n$$

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[2] Although the Geometry of Interaction was presented in terms of partial isometries, the partial isometries used were of a very special form. Precisely, all the action takes place within the image of Barr's injective functor from Partial Injections into Hilbert spaces, $l_2 : \mathbf{pInj}^{op} \to \mathbf{Hilb}$. Thus, although partial isometries were used, all initial and final projectors commute — this is clearly a very restricted case!

For arbitrary $n \geq 1$, $[(ML)(ML)^*]^n (ML) = (MLL^*M^*)^n ML$, and rebracketing gives $[(ML)(ML)^*]^n (ML) = M(LL^*M^*M)^n L = M(F_L E_M)^n L$. Hence

$$lim_{n\rightarrow\infty} [(ML)(ML)^*]^n (ML) = lim_{n\rightarrow\infty} M(F_L E_M)^n L = M(E_M \wedge F_L)L$$

As $E_M \wedge F_L$ is a projector that commutes with both E_M and F_L, it follows from Proposition 5 that

(1) $(E_M \wedge F_L)L : H_1 \rightarrow H_2$ is a partial isometry,
(2) $M(E_M \wedge F_L) : H_1 \rightarrow H_2$ is a partial isometry,
(3) and hence $M(E_M \wedge F_L)L : H_1 \rightarrow H_3$ is a partial isometry.

Finally, when $E_M F_L = F_L E_M$, then $E_M \wedge F_L = E_M F_L = F_L E_M$, and so $M \circ L = ML$. □

Theorem 17 *Partial isometries, with the composition given above, form an inverse category.*

PROOF. It is shown in [46] that the set of all partial isometries acting on a single space, together with this composition is a monoid. The extension to the many-object case is immediate. □ □

4.0.0.4 Notation We denote the category of partial isometries with the above composition, by **pIsom**. By contrast, we denote the category of continuous linear maps on Hilbert spaces (with the usual composition) by **Hilb**. We will using the notation $\circ$ for the composition in **pIsom**, and simply use concatenation to denote the composition in **Hilb**.

Not only is the category **pIsom** closed under the composition $\circ$, but there is a very strong sense in which it is the 'largest' partial isometry satisfying properties $(1)-(3)$ of Lemma 16. Hence, the composition given can be thought of as a supremum within the HML partial ordering, as follows :

Proposition 18 *Let $M : H_2 \rightarrow H_3$ and $L : H_1 \rightarrow H_2$ be partial isometries, and let $P : H_2 \rightarrow H_2$ be a projector such that*

$$MP : H_2 \rightarrow H_3 \quad \textit{and} \quad PL : H_1 \rightarrow H_2 \quad \textit{and} \quad MPL : H_1 \rightarrow H_3$$

are all partial isometries. Then $MPL \leqq M \circ L$, where $\leqq$ is the Halmos-McLaughlin partial order of Definition 7.

PROOF. Since MP and PL are partial isometries, from Proposition 5, $PE_M = E_M P$ and $PF_L = F_L P$. Hence $Q = E_M P F_L$ is a projector satisfying $MPL =$

MQL. We now work with this projector Q. By construction, $Q \leq E_M$ and $Q \leq F_L$, so by definition $Q \leq (E_M \wedge F_L)$. Now consider the unique projectors $Q', R : H_1 \rightarrow H_1$ satisfying $MQL = MLQ'$ and $M(E_M \wedge F_L)L = MLR$ given as in Proposition 11. From Lemma 12, we deduce $Q' \leq R$, so $Q'R = Q'$, and so $MLRQ' = MLQ'$. However, by definition of the Halmos-McLaughlin partial order,

$$MPL = MQL = MLQ' \leqq MLR = M(E_M \wedge F_L)L$$

as required □

It is then straightforward to write down the initial and final projectors of $M \circ L$. These are closely related to the notion of 'pushing a projector through a partial isometry', as given in Proposition 11.

Corollary 19 *Let $L : H_1 \rightarrow H_2$ and $M : H_2 \rightarrow H_3$ be as above. Then*

(1) The initial projector of $M \circ L$ is the unique projector $P \leq E_L$ satisfying $(E_M \wedge F_L)L = LP$, as in Proposition 11.

(2) The final projector of $M \circ L$ is the unique projector $Q \leq F_M$ satisfying $QM = M(E_M \wedge F_L)$, as in Proposition 11.

PROOF. The initial projector P of $M(E_M \wedge F_L)L$ may be given explicitly by :

$$P = L^*(E_M \wedge F_L)M^*M(E_M \wedge F_L)L = L^*(E_M \wedge F_L)E_M(E_M \wedge F_L)L$$

However, $(E_M \wedge F_L) \leq E_M$ and $(E_M \wedge F_L)^2 = E_M \wedge F_L$, so $P = L^*(E_M \wedge F_L)L$. The final projector Q may similarly be shown to be $Q = M(E_M \wedge F_L)M^*$.

Results (1) and (2) then follow by comparing these explicit forms with Proposition 11. □

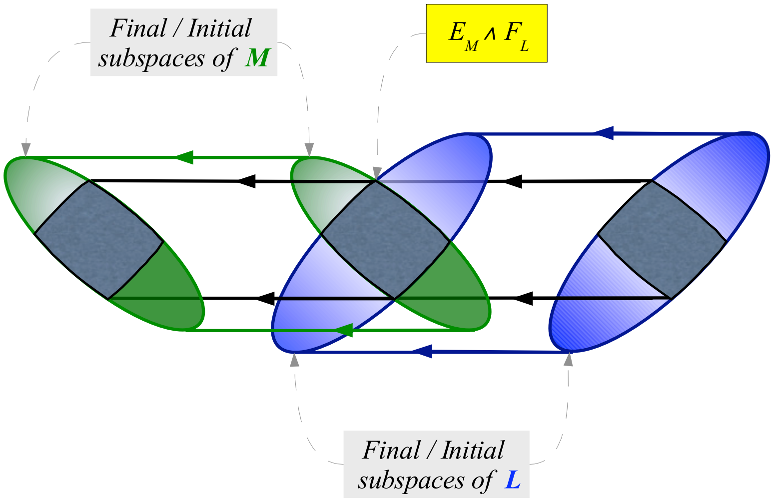
5 The inverse structure of pIsom

From the characterisation of the composition given in Proposition 16, and the inverse category - theoretic result of 17, there is a simple graphical representation of the composition $\circ$, as shown in Figure 5.

We now consider the consequences of the inverse structure on partial isometries. A simple corollary is the commutativity of projectors, as follows :

Corollary 20

Fig. 2. The composition $M \circ L$ of partial isometries



(1) The idempotents of **pIsom** *are exactly the projectors of* **Hilb***.*
(2) All projectors (at the same object) commute, so $E \circ F = F \circ E$*.*
(3) Given projectors $E, F : H \rightarrow H$*, then* $E \circ F = E \wedge F$*.*

PROOF.

(1) Consider an idempotent $L \circ L = L$ in **pIsom**. Then $L = L(E_L \wedge F_L)L$, and hence $E_L \leq F_L$ and $F_L \leq E_L$. Therefore, $E_L = F_L$, and so $L \circ L = LL$, and L is thus idempotent in the category **Hilb**. Hence, as L is a partial isometry it is Hermitian, and so L is a projector.
(2) The commutativity of idempotents follows from the uniqueness of generalised inverses, as standard result of inverse semigroup theory [50].
(3) This is immediate from the definition of $\circ$.

□

5.0.0.5 Commutativity of Projectors The commutativity of projectors in **pIsom** follows from the fact that it is an inverse category — thus the commutativity of projectors is an essential feature of this category. We note the strong distinction with the 'matrix mechanics' formulation of quantum mechanics [] and the behaviour of projectors in the category **Hilb**, where the *non-commutativity of projectors* captures the 'non-classical' behaviour of observations. We discuss this further in Section **??**.

So far, we have seen that the Halmos-McLaughlin partial order generalises the 'quantum logic' ordering of projectors on Hilbert space (Definition 7), and have used the conjunction operation of quantum logic to define a composition on partial isometries (Definition 15 and Lemma 16). This composition is then exactly the calculation of a supremum within the HML partial order (Proposition 18), and allows us to define an inverse category of partial isometries (Theorem 17).

However, every inverse semigroup has a partial order defined on its elements, as in [46], and this has an immediate generalisation to the hom-sets of inverse categories :

Definition 21 The natural partial order of an inverse category
Let S be an inverse semigroup. The **natural partial order** $\trianglelefteq$ *on S is defined by*

$$s \trianglelefteq t \ \Leftrightarrow \ \exists \ e^2 = e \ : \ s = te$$

We refer to [46] for proof that this is indeed a partial order, and its properties.

The generalisation of this notion to inverse categories is immediate. Let $\mathcal{C}$ be an inverse category, and consider $f, g \in \mathcal{C}(X,Y)$. We extend the above definition in the obvious way, so

$$f \trianglelefteq g \ \Leftrightarrow \ \exists \ e^2 = e \in \mathcal{C}(X,X) \ : \ f = ge$$

and this makes the hom-set $\mathcal{C}(X,Y)$ into a partial order.

5.1.0.6 The natural partial order, and the generalised inverse We emphasise that the generalised inverse is *not* an orthocomplement on the partially ordered hom-sets of an inverse category. In particular, given $f \trianglelefteq g \in \mathcal{C}(X,Y)$, then $f^{-1} \trianglelefteq g^{-1} \in \mathcal{C}(Y,X)$. We refer to [46] for a comprehensive list of properties of $\trianglelefteq$ from a semigroup-theoretic point of view. From a categorical point of view, we refer to Section **??**, where the generalised inverse operation $(\_)^{-1}$ is the *dual* operator in a compact closed category.

Theorem 22 *The Halmos-McLaughlin partial order $\leqq$ is exactly the natural partial order $\trianglelefteq$ on the inverse category* **pIsom**.

PROOF. Consider $L, M \in$ **pIsom**(H,K).

- ($\Rightarrow$) Assume $L \leqq K$. Then by definition, $L = ME_L$. Hence, as ME_L is a partial isometry, $ME_L = M(E_M \wedge E_L) = M \circ E_L$, so $L \trianglelefteq M$.

- (⇐) Assume $L \trianglelefteq M$. Then there exists some $G^2 = G \in \mathbf{pIsom}(H,H)$ such that $L = M \circ G$. Therefore, $L \circ G = M \circ G \circ G = M \circ G = L$, and the initial subspace of L is a subspace of the initial subspace of M. Hence $L \leqq M$, as required.

□ □

6 Compact closure, teleportation, and partial isometries

The category of partial isometries can reasonably be considered as a categorification of von Neumann - Birkhoff quantum logic. However, from Sections 5.0.0.5 and 3.4.0.3, we do not expect to be able to describe all quantum operations within the category of partial isometries.

We now consider whether a (post-selected version of) the teleportation protocol [8] can be expressed in **pIsom**. We approach this question categorically, via the alternative *categorical foundations for QM* given in [5,6]. In these categorical foundations, the teleportation protocol is taken as primitive, and an illustration of compact closure (as described in 6.2 below). We therefore consider whether the category of partial isometries is compact closed.

We conclude that it is not, but the reasons for failure of compact closure are significant.

6.1 Compact closed categories

Compact closed categories are symmetric monoidal closed categories where the categorical closure is of a particularly well-behaved form. We refer to [47] for closed categories generally, and [42,43] for logical interpretations. Compact closed categories, and a coherence theorem are described in [**?**], and a construction of compact closed categories from traced monoidal categories is given in [2] and [41]. Finally, we refer to [2,33,35,4] for logical and computational interpretations, and [34] for one-object, or untyped, compact closed categories.

Definition 23 compact closed categories, dual on objects, dual on arrows
Let $\mathbf{C}, \otimes$ be a symmetric monoidal category, with unit object I, so for all $A \in Ob(\mathcal{C})$, there exists isomorphisms

$$\rho_A : A \otimes I \to A \quad and \quad \lambda : I \otimes A \to A$$

The category $\mathbf{C}$ *is called* **compact closed** *when, for all* $A \in Ob(\mathbf{C})$*, there exists a* **dual object** A^**, together with distinguished arrows*

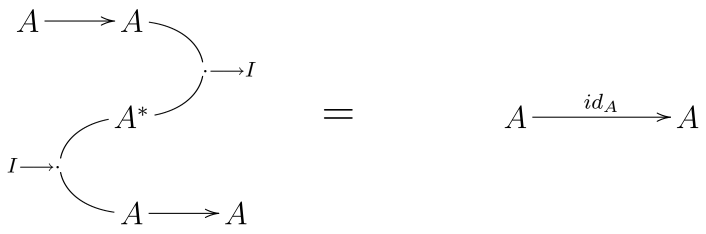
- *The unit arrow* $\epsilon_A : A \otimes A^* \to I$
- *The counit arrow* $\eta_A : I \to A^* \otimes A$

that satisfy

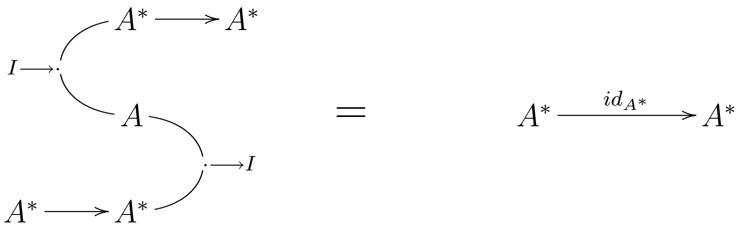
$$\lambda(\epsilon_A \otimes 1_A)(1_A \otimes \eta_A)\rho^{-1} = 1_A \quad \textit{and dually,} \quad \rho_{A^*}(1_{A^*} \otimes \epsilon_A)(\eta_A \otimes 1_{A^*})\lambda_{A^*}^{-1} = 1_{A^*}$$

Using the diagrammatic notation of [40,41], this may be drawn as in Figure 23. Note that (following the usual convention) this diagram omits the unit object isomorphisms $A \cong A \otimes I$*, &c.*

Fig. 3. Axioms for compact closure



and dually,



The dual operation on objects $(\_))^*$*, together with the unit and counit arrows may be used to define the* **dual on arrows**. *Given* $f \in \mathcal{C}(A, B)$*, then* $F^* \in \mathcal{C}(B^*, A^*)$ *is defined by*

$$f^* = (1_{A^*} \otimes \epsilon_B)(1_{A^*} \otimes f \otimes 1_{B^*})(\eta_A \otimes 1_{B^*} : B^* \to A^*$$

Diagrammatically, this is as shown in Figure 23.

Note that in a compact closed category, the arrows η_A *and* μ_A *are dual, so* $\eta_A^* = \epsilon_A$.

Fig. 4. The dual operation on arrows

$$\begin{tikzcd}
 & & A^* \arrow[rr, "1_{A^*}"] & & A^* & \\
I \arrow[r] & \cdot \arrow[ur, bend left, no head] \arrow[dr, bend right, no head] & & & & \\
 & & A \arrow[r, "f"] & B \arrow[dr, bend left, no head] & & \\
 & & & & \cdot \arrow[r] & I \\
B^* \arrow[rrr, "1_{B^*}"'] & & & B^* \arrow[ur, bend right, no head] & &
\end{tikzcd}$$

6.2 *Categorical foundations of quantum mechanics*

In the categorical foundations of quantum mechanics [5,6], the teleportation protocol [8], and the notion of teleportation as computation [12] are taken as primitive.

The interpretation is based on a compact closed category (with additional structure), where the dual operation is the identity on objects, so $H^* = H$. The axiom for compact closure shown in Figure **??** below is interpreted as follows : The monoidal tensor is the formation of a composite system (i.e. the usual tensor product of Hilbert space), and the trivial object I is a one-dimensional space, so $I \otimes H \cong H \cong H \otimes I$.

Fig. 5. Compact closure with self-dual objects

$$\begin{tikzcd}
H \arrow[r] & H \arrow[dr, bend left, no head] & & & & \\
 & & \cdot \arrow[r] & I & & \\
 & H \arrow[ur, bend right, no head] & & = & H \arrow[r, "id_H"] & H \\
I \arrow[r] & \cdot \arrow[u, bend left, no head] \arrow[d, bend right, no head] & & & & \\
 & H \arrow[r] & H & & &
\end{tikzcd}$$

The counit arrow $\eta_H : I \rightarrow H \otimes H$ is interpreted as the creation of a maximally entangled bell state $|\mathcal{B}ell\rangle$ and its dual $\epsilon_H : H \otimes H \rightarrow I$ is a (post-selected) measurement against the same maximally entangled state. The requirement that the composite shown on the l.h.s. of Figure 6.2 is exactly the identity gives the teleportation protocol (post-selected) of [8].

In the concrete category of finite-dimensional Hilbert spaces, consider an n-dimensional space H, with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$. The maximally entangled Bell state is given by $\mathcal{B}ell = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathbf{e}_j \otimes \mathbf{e}_j$, so the corresponding unit and counit maps are $\langle \mathcal{B}ell| : H \rightarrow \mathbb{C}$ and $|\mathcal{B}ell\rangle : \mathbb{C} \rightarrow H$. Direct calculation gives (up to the appropriate renormalisation, corresponding to the post-selection of measurement outcome) $(\langle \mathcal{B}ell| \otimes 1_H)(1_H \otimes |\mathcal{B}ell\rangle) = 1_H$.

6.3 Logical interpretations

Although the foundations for quantum mechanics of Section 6.2 above are presented categorically, rather than logically, the fact that a monoidal closed (in fact, compact closed) category is key to this system leaves it open to a logical interpretation. An explicit logical interpretation is given in [17,6,16], by analogy with the connectives and structure of linear logic.

The application of compact closure to logical systems arose from analyses of Girard's Geometry of Interaction system [22,23], representation of Linear Logic [21]. We refer to [10] for a good historical overview. The categorical interpretation of Linear Logic as a whole is in terms of $*$-autonomous categories. However, the Geometry of Interaction system gave a representation of a restricted fragment (the *multiplicatives*). It was also degenerate (and hence a *representation*, rather than a *model*, of this fragment of linear logic) in that the conjunction and disjunction are identified (a case is made in [4] that the correct interpretation of the geometry of interaction system is as *combinatory logic*).

The logical interpretation of the categorical analysis of Section 6.2 above is even more degenerate, in that all objects are *self-dual*. In $*$-autonomous and compact closed categories, the dual $*$ on objects is interpreted as logical negation. However, there remains a monoidal tensor and (via the monoidal closure) an internal hom functor that satisfy the required properties for a logical system.

We refer to [16,6] for more details.

6.4 Is **pIsom** *compact closed ?*

Before answering the above question, we need to explain why (apart from wishful thinking) we might think that $(\mathbf{pIsom}, \otimes)$ should be compact closed – at least, in the finite-dimensional case. A suggestive, but incorrect, train of thought is as follows :

Non-theorem 24 *Consider the defining identity of compact closure (in the self-dual case) :*

$$\lambda(\epsilon_A \otimes 1_A)(1_A \otimes \eta_A)\rho^{-1} = 1_A$$

If, as in the categorical foundations described in Section 6.2, we interpret the counit η_H and unit ϵ_H as the bra and ket $|\mathcal{B}ell\rangle : \mathbb{C} \to H$ and $\langle \mathcal{B}ell| : H \to \mathbb{C}$, we observe that these are partial isometries. Hence, as partial isometries are closed under the tensor product, $(\epsilon_A \otimes 1_A)$ and $(1_A \otimes \eta_A)$ are both partial isometries. The identities isomorphisms $\rho^{-1} : H \cong H \otimes \mathbb{C}$ and $\lambda : \mathbb{C} \otimes H \cong H$ are trivially unitary, and so (by Corollary 6) the linear maps

$$\mathcal{T}ele = \lambda(\epsilon_A \otimes 1_A) \quad \textit{and} \quad \mathcal{P}ort = (1_A \otimes \eta_A)\rho^{-1}$$

are both partial isometries.

Based on the categorical foundations, we wish to claim that the composite $\mathcal{T}ele\mathcal{P}ort : H \to H$ is the identity map (again, a partial isometry). From Theorem 5, the composite KL of two partial isometries is itself a partial isometry exactly when the final projector of L commutes with the initial projector of K, in which case (by Lemma 16) $K \circ L = KL$.

We thus wish to conclude that $\mathcal{T}ele \circ \mathcal{P}ort = \mathcal{T}ele\mathcal{P}ort = 1_H$.

To see that this reasoning is incorrect, we explicitly exhibit the final projector of $\mathcal{P}ort$, and the initial projector of $\mathcal{T}ele$, and show that these do <u>not</u> commute.

Lemma 25 *Consider the partial isometries $\mathcal{T}ele$ and $\mathcal{P}ort$, defined in Non-Theorem 24 above. Then the initial and final projectors $E_{\mathcal{T}ele}$ and $F_{\mathcal{P}ort}$ do not commute, so $\mathcal{T}ele \circ \mathcal{P}ort \neq \mathcal{T}ele\mathcal{P}ort$.*

PROOF. Consider a complex N-dimensional space H, with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$. The maximally entangled Bell state is given by $\mathcal{B}ell = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathbf{e}_i \otimes \mathbf{e}_i$, so the unit and counit maps are

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} |\mathbf{e}_j \mathbf{e}_j\rangle \quad \text{and} \quad \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \langle \mathbf{e}_k \mathbf{e}_k|$$

respectively. From this, the partial isometries $\mathcal{T}ele : H \otimes H \otimes H \to H$ and $\mathcal{P}ort : H \to H \otimes H \otimes H$ may be given explicitly by

$$\mathcal{T}ele = \frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} |\mathbf{e}_a\rangle\langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a| \quad \text{and} \quad \mathcal{P}ort = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_\mathbf{j}\rangle\langle \mathbf{e}_\mathbf{i}|$$

Thus, the final projector of $\mathcal{P}ort$ may be given by

$$F_{\mathcal{P}ort} = \left(\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_\mathbf{j}\rangle\langle \mathbf{e}_\mathbf{i}| \right) \left(\frac{1}{\sqrt{N}} \sum_{k,l=1}^{N} |\mathbf{e}_k \mathbf{e}_l \mathbf{e}_l\rangle\langle \mathbf{e}_k| \right)^*$$

$$= \frac{1}{N} \sum_{i,j,k,l=1}^{N} |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle \delta_{ik} \langle \mathbf{e}_k \mathbf{e}_l \mathbf{e}_l| = \frac{1}{N} \sum_{i,j,l=1}^{N} |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle \langle \mathbf{e}_i \mathbf{e}_l \mathbf{e}_l|$$

Similarly, the initial projector of $\mathcal{T}ele$ is given by

$$E_{\mathcal{T}ele} = \left(\frac{1}{\sqrt{N}} \sum_{c,d=1}^{N} |\mathbf{e}_d \mathbf{e}_d \mathbf{e}_c\rangle \langle \mathbf{e}_c| \right) \left(\frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} |\mathbf{e}_a\rangle \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a| \right)$$

$$= \frac{1}{N} \sum_{a,b,c,d=1}^{N} |\mathbf{e}_d \mathbf{e}_d \mathbf{e}_c\rangle \delta_{ac} \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a| = \frac{1}{N} \sum_{a,b,d=1}^{N} |\mathbf{e}_d \mathbf{e}_d \mathbf{e}_a\rangle \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a|$$

Straightforward direct calculation will verify that $E_{\mathcal{T}ele} F_{\mathcal{P}ort} \neq F_{\mathcal{P}ort} E_{\mathcal{T}ele}$ and so by Theorem 5, $\mathcal{T}ele \circ \mathcal{P}ort \neq \mathcal{T}ele\mathcal{P}ort$. □

What, then has gone wrong in the reasoning in Non-Theorem 24 ? Our claim is that the renormalisation, or implicit post-selection, in Section 6.2 is incompatible with a study of teleportation via partial isometries, as follows :

Theorem 26 *Consider the partial isometries $\mathcal{T}ele : H \to H \otimes H \otimes H$ and $\mathcal{P}ort : H \otimes H \otimes H \to H$ of Non-Theorem 24. Their composite, as linear maps, is $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N} 1_H$ and therefore their composite in the category* **pIsom** *is $\mathcal{T}ele \circ \mathcal{P}ort = 0_H$.*

PROOF. By direct calculation,

$$\mathcal{T}ele\mathcal{P}ort = \frac{1}{N} \sum_{a,b,i,j=1}^{N} |\mathbf{e}_a\rangle \langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a | \mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle \langle \mathbf{e}_i|$$

$$= \frac{1}{N} \sum_{a,b,i,j=1}^{N} |\mathbf{e}_a\rangle \delta_{bi} \delta_{bj} \delta_{aj} \langle \mathbf{e}_i| = \frac{1}{N} \sum_{a=1}^{N} |\mathbf{e}_a\rangle \langle \mathbf{e}_a| = \frac{1}{N} Id_H$$

Using the original definition of composition in **pIsom**,

$$\mathcal{T}ele \circ \mathcal{P}ort = lim_{K \to \infty} \left[(\mathcal{T}ele\mathcal{P}ort)(\mathcal{T}ele\mathcal{P}ort)^* \right]^K \mathcal{T}ele\mathcal{P}ort$$

As $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N} Id_H$ is self-adjoint,

$$\mathcal{T}ele \circ \mathcal{P}ort = lim_{K \to \infty} \left[\frac{1}{N^2} Id_H \right]^K \frac{1}{N} Id_H = 0_H$$

□

6.4.0.7 Post-selection and partial isometries

A direct calculation gives that, as a straightforward composite of linear maps, $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N} Id_H$. This extraneous factor of $\frac{1}{N}$ occurs simply because the probability of observing this particular experimental outcome (i.e. the Bell state $\mathcal{B}ell = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathbf{e}_j \otimes \mathbf{e}_j$) is exactly $\frac{1}{N}$.

In the categorical foundations approach, post-selection of measurement outcomes is used to eliminate this scaling factor, and hence give compact closure. Unfortunately, this does not appear to be a possibility in **pIsom**. As the unit and co-unit are each others dual they must have the same norm. Thus we cannot simply 'scale' the unit map by a factor of N — we need to scale the unit and co-unit equally by a factor of $\sqrt{N}$. However, all partial isometries (except the zero map) have operator norm of 1, and such a 'scaled' unit / co-unit pair would no longer be members of the category.

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