Planar two-way automata

finite state machines and the Temperley-Lieb algebra

–Work in Progress–

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Project GR/S82176/01

Slides available at : *http://www-users.cs.york.ac.uk/∼phines/fountain.pdf*

Historical Context (I)

(1959) **J. Sheperdson** introduces & studies **two-way automata**.

- *Significant result :* 2-way automata recognise the *regular languages*.
- This is the same class as one-way automata (FSA).
- This destroys interest in 2-way automata as computing devices for many years ...

(1986) **V. Jones** introduces a **polynomial knot invariant**.

- This is based on a von Neumann algebra called the *Temperley-Lieb algebra*.
- Computing it is a difficult task – it is a $\#P$ complete problem.
- It is still (much) simpler that the decision procedure for knot equivalence.

Historical Context (II)

(1989) **J.-C. Birget** treats 2-way automata as **transductions**.

- This approach is
  1. not immediately about recognising languages.
  2. trivial, for 1-way automata.
- *Significant result :* There is a bizzare notion of *composition* that models concatenation of inputs.

(1987 - 1994) **J.-Y. Girard** introduces **Linear Logic**, and the **Geometry of Interaction** program.

- *Linear Logic* is a resource-sensitive, reversible, decomposition of classical logic.
- The *Geometry of Interaction* papers (I,II) aim to model linear logic.
  - These aim to “treat logic as *dynamic*, not *static*”.
  - However, they are degenerate models, and identify conjuction and disjunction.

Historical Context (III)

(1996) **A. Joyal, R. Street, D. Verity** give the $\mathbf{Int}$ construction.

- A categorical construction derived from knot theory, braid closure, and feedback.

(1996) (Independently) **S. Abramsky** gives the $\mathbf{GoI}$ construction.

- this is a categorical construction based on Girard's Geometry of Interaction papers.

These two constructions are the same – at least, in the symmetric case.

(1994-1998) **PMH** writing **PhD Thesis** ...

- Observes that Girard's system is based on the $\mathbf{Int}$ construction, applied to *reversible operations*.

Historical Context (IV)

(1996 or so) **M. V. Lawson** (supervising PMH) considers **Birget's Bizzare Composition**.

- This is an example of the $\mathbf{Int}$, or $\mathbf{GoI}$ construction.
- This observation is duly expanded & made into a chapter of PMH's thesis.
- A later extension (not in Thesis) gives models of Space-bounded Turing machines in the same terms.

(1997) **J. Watrous** describes **quantum 2-way automata**.

- These are shown to recognise some *context-sensitive languages*, so are more powerful than either classical or quantum one-way automata.
- However, his formulation does not allow for analogues of B.B.C.

Historical Context (V)

(1999) A paper describing two-way automata & space-bounded Turing machines in terms of the Geometry of Interaction is rejected, because *"Automata theory is all about recognising languages"*.

(2002) **S. Abramsky, P. Scott, E. Haghverdi** study GoI and **combinatory logic**

- They show Girard's system is a complete model of Combinatory Logic (& hence λ-calculus).

(2002-present) **S. Abramsky, B. Coecke** consider **entanglement** & **measurement**, categorically.

- This leads to their **categorical foundations** for quantum mechanics, based on GoI.
- This is a **different form** of GoI, based on fixed points / quantification, rather than iteration.
- There is a 'parallel history' of this form – the two strands are often called *wave-style* and *particle-style*.

Historical Context (VI)

(2003) The paper on **GoI**, two-way automata, and bounded Turing machines finally appears – in a more prestigious journal (with more open-minded referees).

(2004-2005) **PMH, P. Scott** consider GoI on **Hilbert spaces** & **unitary maps**.

- Measurement is *excluded*, but iteration & superposition are all-important.
- This is *particle-style* GoI, unlike Abramsky - Coecke's categorical foundations for QM.

(2005) **S. Abramsky** describes the **Temperley-Lieb algebra** in terms of the GoI construction.

- This axiomatises planar diagrams – i.e. those where transitions may be drawn without crossings.
- He also gives a **logical interpretation** – the "*planar lambda calculus*".
- This is also *particle-style* GoI.

Historical Context (VII)

(2006) **D. Aharanov, V. Jones, Z. Landau** give a **quantum algorithm** for the Jones polynomial.

- This computes the Jones Polynomial at primitive roots of unity, in polynomial time.
- This is a provable exponential speedup on classical computation.
- ... however, the whole polynomial cannot be reconstructed from its value at roots of unity.

(2006) **The Kats, Kets and Cloisers conference (Oxford)**

- Talks are given on most of the above subjects ... leading to many questions!

Some questions ...

1. There is an implicit connection between the T.-L. algebra and two-way automata :

   – can this be made explicit ?

2. (This requires :) what does planarity mean for two-way automata ?

3. What is the complexity class of the resulting machines ?

4. Is there a connection with :

   (a) quantum two-way automata ?

   (b) knot theory ?

   (c) The Jones polynomial algorithm ?

Back to Basics (I) – 2-way automata

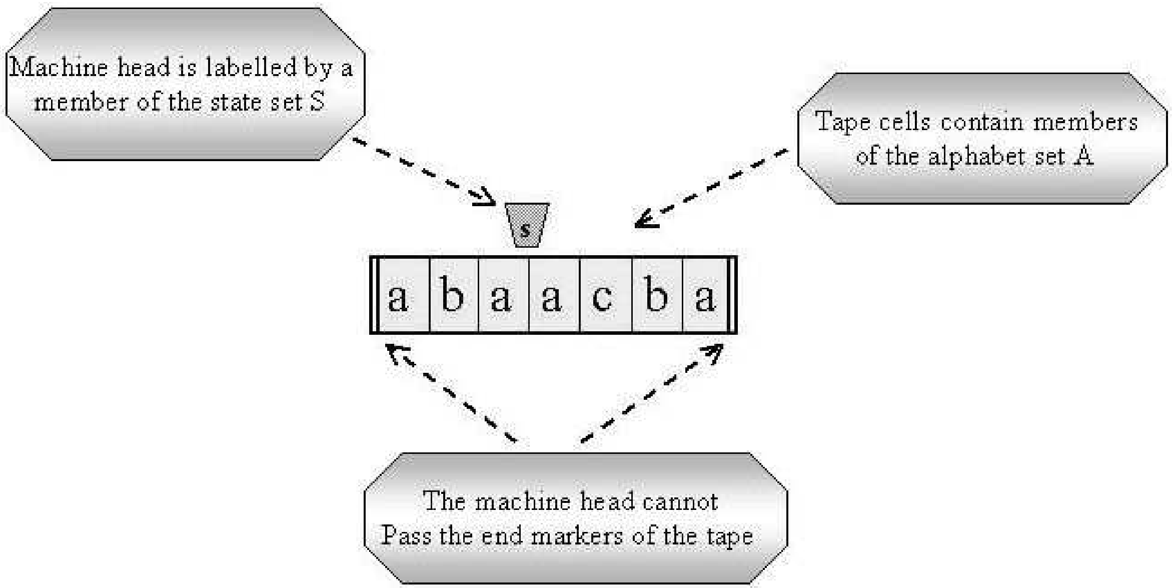
A two-way automaton is specified by :

- A set A of **Alphabet Symbols**
- A set S of **States**
  - S is divided into **left-moving states** L, and **right-moving states** R, so $S = L \uplus R$.
- For each $a \in A$, a **transition function** $[a] : S \to S$.

As a state machine, there is :

- A **finite tape**, with alphabet symbols written on it.
- A single **machine head**, labelled by a state.
- **End markers** for the tape.

The anatomy of a 2-way automaton



This is the definition proposed by J.-C. Birget. It differs from :

1. Sheperdson's model : the machine head points at cell boundaries, rather than cells themselves.

2. Watrous' model : it has end markers, rather than a circular tape.

The dynamics of a 2-way automaton

At each primitive step :

1. If the machine head has a *left-moving* label, it moves onto the cell to the *left*.

   – alternatively, it moves onto the cell to the *right*.

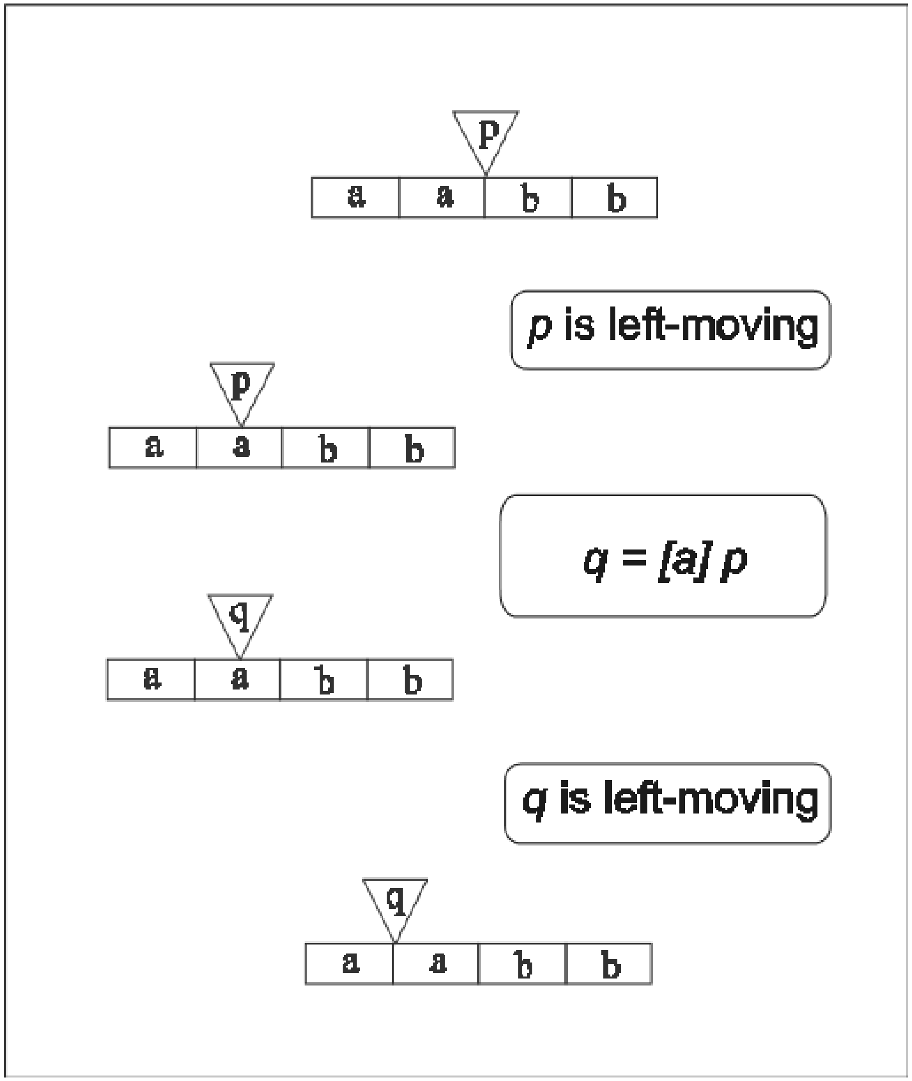
2. The cell contents determine a new label for the machine head.

3. If the new label is *left-moving*, the machine head moves to the *left* of the cell.

   – alternatively, it moves to the *right* of the cell.

(This description is due to PMH. It is simpler than, but equivalent to, Birget's definition – as shown in 2003 paper).

An example 2-way automaton computation:



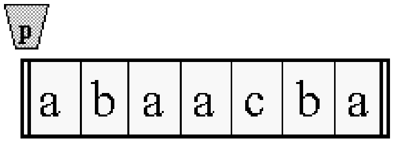
Boundary configurations

A **configuration** is simply an instantaneous description of a 2-way automaton.

From the definition — a configuration with the machine head over an end-marker has either

1. no *'previous configuration'* under the machine evolution.
2. no *'next configuration'* under the machine evolution.

Call such configurations the **boundary configurations**.



Note : J. Watrous' quantum two-way automaton has *no boundary configurations* !

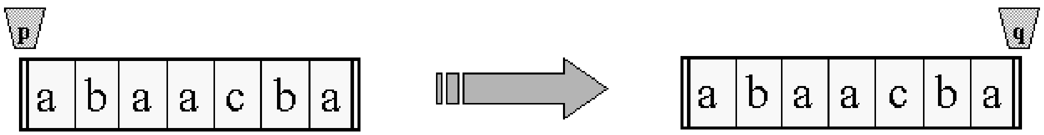
Birget's Relations

Each word $w \in A^*$ determines a relation $[w]$ on the state set.

q is related to p by $[w]$, written $q[w]p$ exactly when :

There exists a boundary-to-boundary computation that

1. Starts with p labelling the machine head.
2. Finishes with q labelling the machine head.



This is called the **global transition relation** of w.

Some basic results :

- The **transition relation** for a singleton $a \in A$ is exactly the **transition function** from the definition.
- If a two-way automaton is **deterministic**, every transition relation is a **partial function**.
- If a 2-way automaton is **reversible**, every transition relation is a **partial bijection**.

A not-so-basic result :

- We can derive $[uv]$ from $[u]$ and $[v]$ separately, using (equivalently)
  - Birget's Bizzare Composition.
  - The $\mathbf{Int}$ or $\mathbf{GoI}$ construction.

Back to Basics (II) – the Temperley-Lieb algebra

- Originally discovered by H. Temperley & E. Lieb, in 1971.
  - Used for statistical models of physics on discrete lattices.
  - Presented as an algebra with Generators & Relations.

Definitions :

The **Temperley-Lieb monoid** M_n has generators $\delta, U_1, U_2, \ldots U_n$, and relations

$$U_i U_j U_i = U_i \quad \text{for all} \quad |i-j| = 1$$

$$U_i^2 = \delta U_i = U_i \delta$$

$$U_i U_j = U_j U_i \quad \text{for all} \quad |i-j| > 1$$

The **Temperley-Lieb algebra** :

- Take the ring L_X of all 1-variable Laurent polynomials over X, and fix some $\tau \in L_X$.
- The T.-L. algebra is the monoid algebra of formal linear combinations $\sum_i l_i m_i$,

$$l_i \in L_X \quad \text{and} \quad m_i \in M_n$$

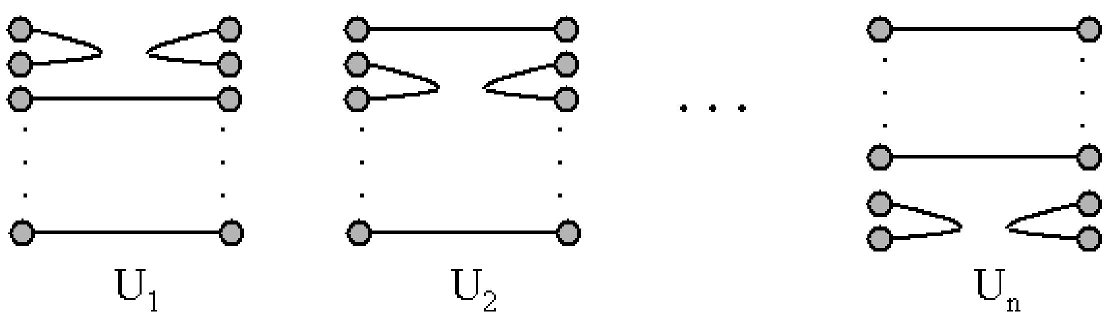
with the quotient $\delta = \tau.1$.

The Temperley-Lieb algebra as planar diagrams

The T.-L. algebra :

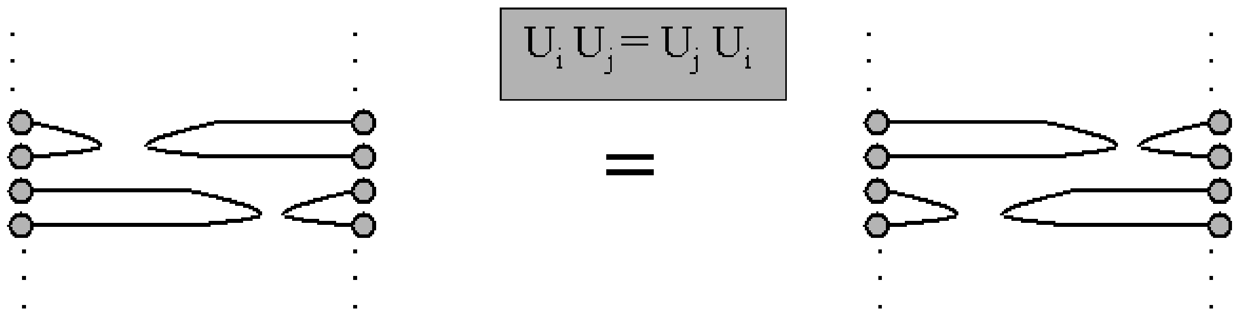
- Independently rediscovered by V. Jones, (in 1985), it plays a starring rôle in his knot polynomial.
- L. Kauffman gave an interpretation (in 1990) as *planar diagrams*.

THE GENERATORS OF THE TEMPERLEY-LIEB MONOID

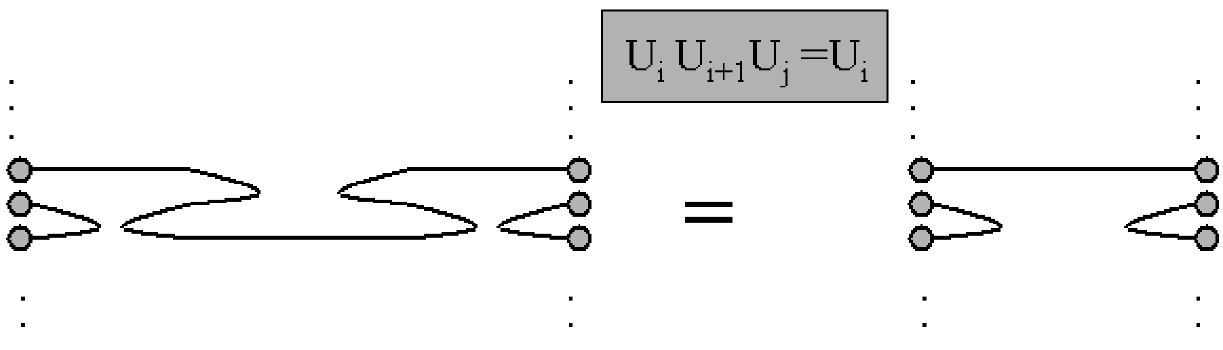


These are considered up to *planar isotopy* — and this provides the relations between generators.

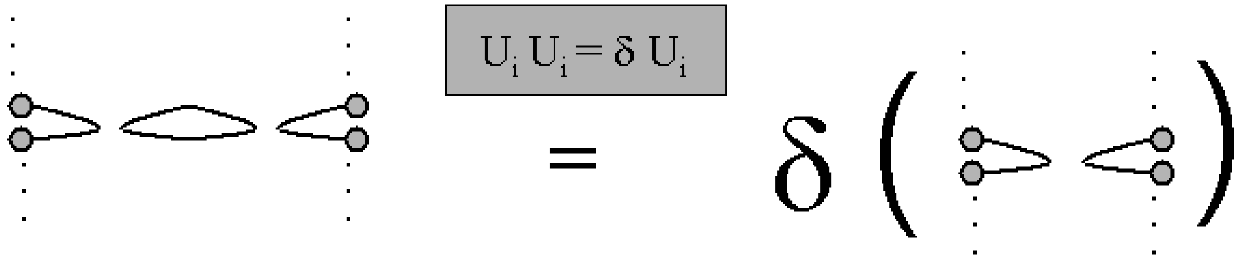
$$U_i\, U_j = U_j\, U_i$$



$$U_i\, U_{i+1} U_j = U_i$$



The only ‘non-obvious’ relation is that closed loops become global scaling factors :



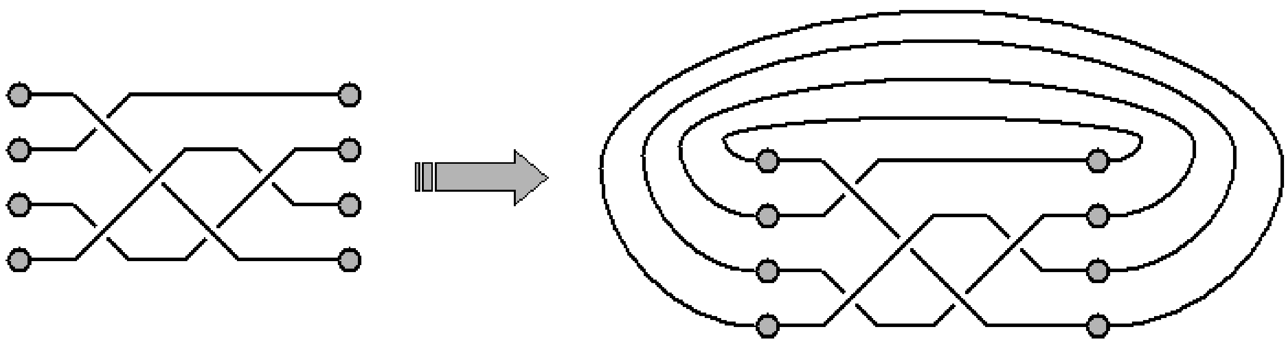
Interesting special cases occur when the scaling factor is :

- $\delta = 0$.
- $\delta = 1$.
- $\delta = \omega$, an n-th root of the identity, so $\omega^n = 1$.

The Temperley-Lieb algebra, and knot theory

The key steps are :

Braid closure A braid diagram may be closed by adding in feedback loops.



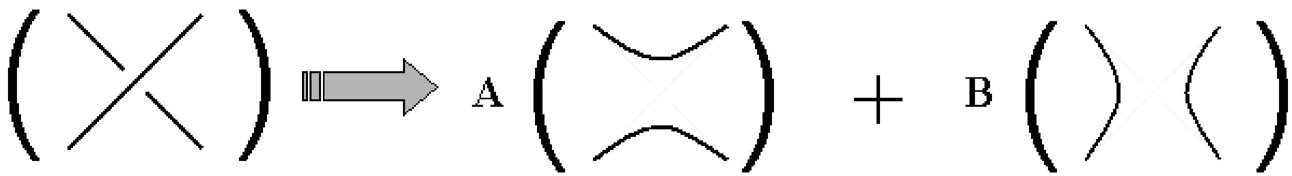
Traditional knot theory – every knot or link is the closure of a braid diagram.

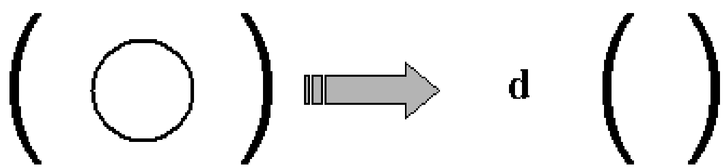
The **Jones polynomial**, as described by **Kauffman**, is computed by ‘eliminating crossings in a diagram’.

A *diagram with crossings* is mapped to the formal sum of *diagrams without crossings*.

The **Jones polynomial**, as described by **Kauffman**, is computed by (recursively)

1. Replacing crossings with weighted formal sums of link diagrams.
2. Replacing unknotted loops with values.





The weights are *Laurent polynomials* over 1 variable, X, and taking

- $A = X$
- $B = X^{-1}$
- $d = -X^2 + X^{-2}$

gives invariance under the Reidemeister moves.

Different views of the Temperley-Lieb monoid

The Temperley-Lieb algebra is the monoid-algebra over a ring of 1-variable Laurent polynomials.

The *relevant monoid* can be seen as :

1. **Knot Theory** – braid diagrams without crossings.
2. **Pure Algebra** – a monoid with generators & relations.
3. **Geometry of Interaction** – a construction involving ordered partial isometries.
4. **Two-way automata** – *a special class of transition relations.*

State-ordered two-way automata

Recall the definition of a two-way automaton :

- A set A of **Alphabet Symbols**
- A set $S = L \uplus R$ of **left-moving** and **right-moving** States.
- For each $a \in A$, a **transition function** $[a] : S \to S$.

We also require :

1. A **partial order** $\leq$ on the state set Q.
2. An **involution** $\sigma : Q \to Q$, so $\sigma^2 = 1_Q$.

These satisfy 2 axioms :

- The subsets L and R are **chains** – i.e. totally ordered subsets.
- The involution σ is **anti-monotonic**,

$$p \leq q \ \Rightarrow \ \sigma(q) \leq \sigma(p)$$

Simple consequences

Simple consequences of this definition :

1. Q has the same number of left- and right- moving states : $|L| = |R|$

2. Both $L, \leq$ and $R, \leq$ are isomorphic to $1, 2, \ldots, n$

3. Left- and right- moving states are *incomparable* : $l \# r$, for all $l \in L, r \in R$.

Some conventions

For 2n states, write the left-moving states as

$$\overleftarrow{1} \leq \overleftarrow{2} \leq \ldots \leq \overleftarrow{n}$$

and the right-moving states as

$$\overrightarrow{1} \geq \overrightarrow{2} \geq \ldots \geq \overrightarrow{n}$$

so

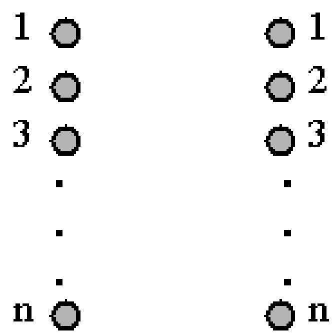
$$\sigma(\overleftarrow{a}) = \overrightarrow{a} \quad \text{and} \quad \sigma(\overrightarrow{a}) = \overleftarrow{a}$$

Transition diagrams

In this setting, there is a natural way of drawing transition relations.

Let $w \in A$ be a word over the input alphabet.

Start with 2 columns of nodes, labelled $1 \ldots n$

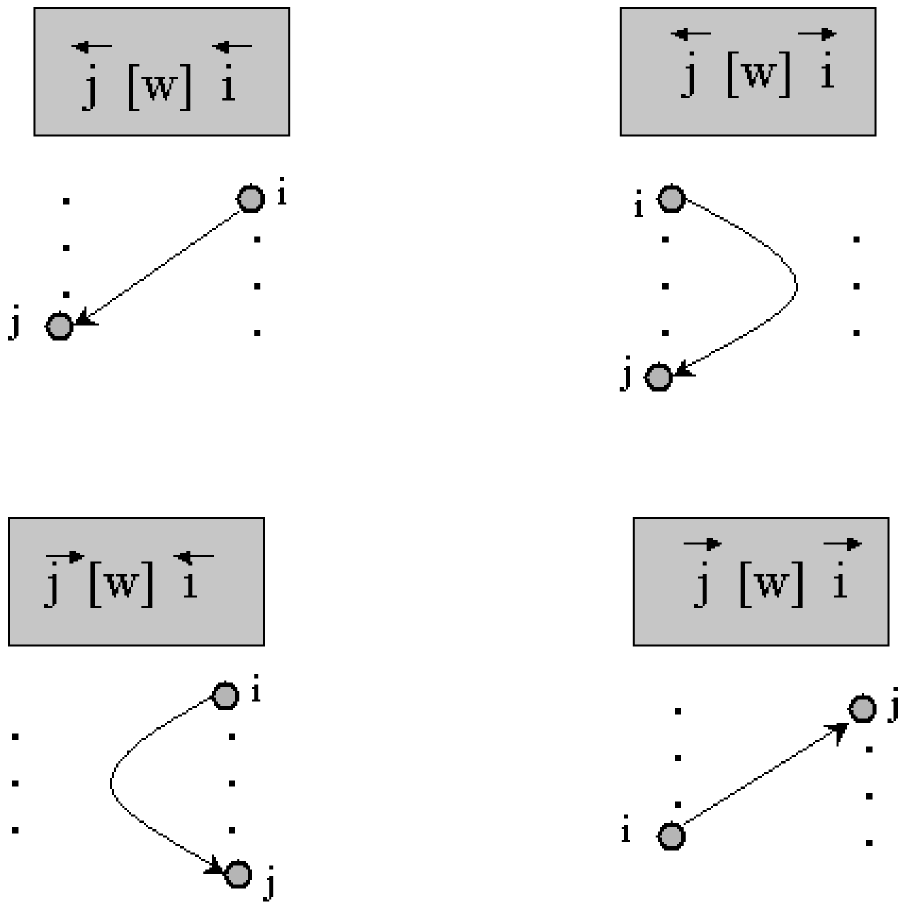


For each pair of states q, p related by the transition relation $[w]$,

$$q[w]p$$

Draw a directed line on this diagram ...

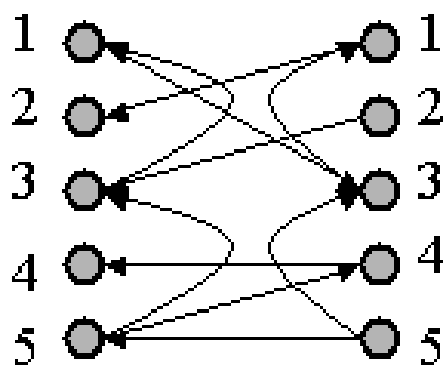
From relations to diagrams :



Each transition relation determines, and is determined by a **transition diagram.**

the T.-L. monoid, and transition diagrams ?

Every transition relation $[w]$ describes a diagram such as



Questions : When are these diagrams

1. **undirected ?**

   i.e. The direction on the arrows does not matter.

2. **planar ?**

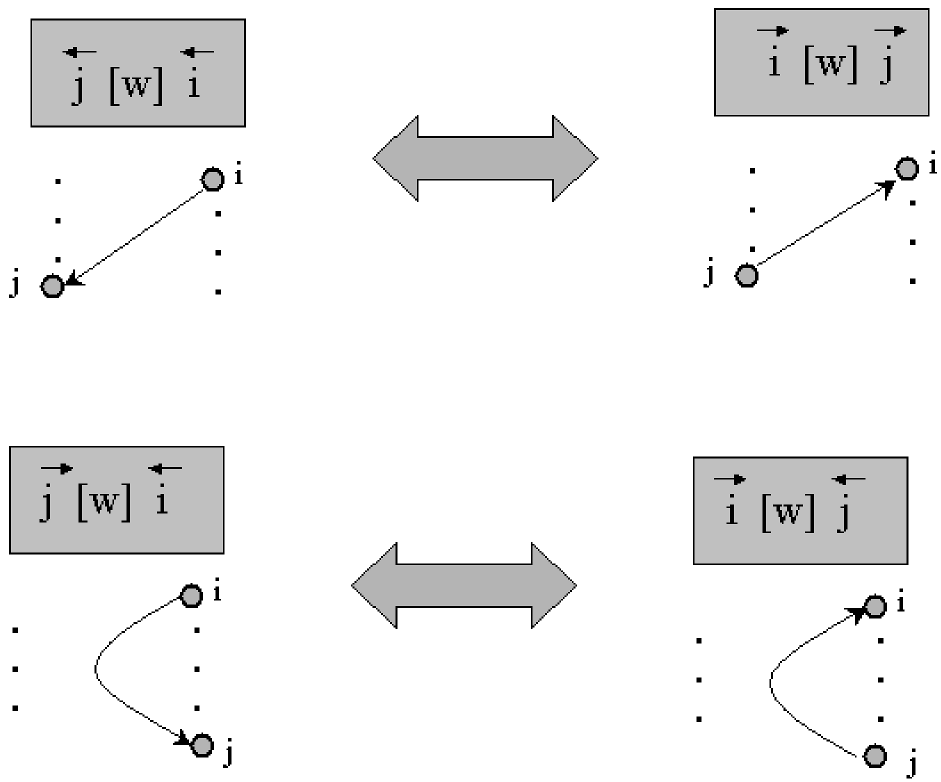
   i.e. Lines in the diagram do not cross.

– the intention is to reproduce Kauffman's diagrammatic presentation of the Temperley-Lieb monoid.

Undirected transition diagrams, graphically

A diagram is <u>undirected</u> when :

- whenever there is a line from node x to node y, there is also a line from node y to node x.



Undirected transition diagrams, algebraically

From the interpretation of diagrams as relations, planarity states :

$$y[w]x \Leftrightarrow \sigma(x)[w]\sigma(y)$$

writing σ in relational form, and using the relational converse,

$$y[w]x \Leftrightarrow y\sigma[w]^c\sigma x$$

Since σ is an involution, this becomes

$$[w] = \sigma^{-1}[w]^c\sigma$$

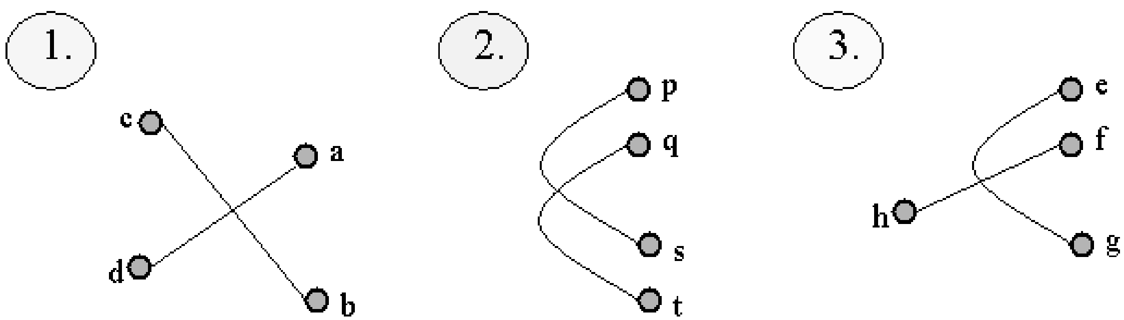
– giving a simple condition for a transition diagram to be undirected.

Enforcing Planarity – diagramatically

Let $w \in A^*$ be an input word, with an undirected transition diagram.

Question when is this planar ??

To enforce planarity we need to rule out 3[a] possibilities :



(Note : each undirected diagram corresponds to 4 statements about the transition relation $[w]$ — a number of cases follow by symmetry.)

[a](Up to left-right and source-target symmetry ...)

Enforcing Planarity – algebraically

Let $w \in A^*$ be an input word, with an undirected transition diagram.

Our claim is :

The following 2 conditions force a transition diagram for w to be planar.

- **Inverse-monotonicity :**

  Given

  $$q[w]p \text{ and } q'[w]p'$$

  then

  $$q \leq q' \Rightarrow p \leq p'$$

- **The interval condition :**

  Given

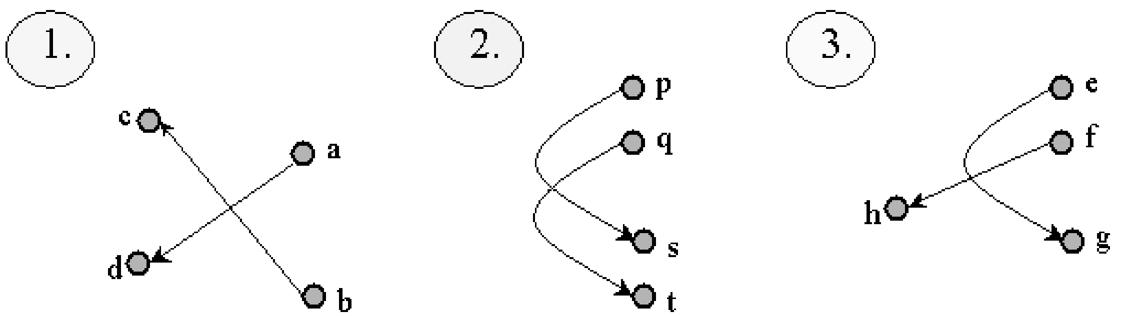
  $$y[w]x \text{ and } b[w]a$$

  then

  $$a \leq x \leq \sigma(b) \Rightarrow \sigma(a) \leq y \leq b$$

Planarity & order theory

Consider 3 distinct crossing types, drawn with an orientation :



From **1.** : $\overleftarrow{d}\,[w]\,\overleftarrow{a}$ and $\overleftarrow{c}\,[w]\,\overleftarrow{b}$. However, $\overleftarrow{a} \leq \overleftarrow{b}$ but $\overleftarrow{c} \geq \overleftarrow{d}$, contradicting **inverse-montonicity**.

From **2.** : $\overrightarrow{s}\,[w]\,\overleftarrow{p}$ and $\overrightarrow{t}\,[w]\,\overleftarrow{q}$. However, $\overleftarrow{p} \leq \overleftarrow{q}$ but $\overrightarrow{s} \geq \overrightarrow{t}$, contradicting **inverse-montonicity**.

From **3.** : $\overleftarrow{e} \leq \overleftarrow{f} \leq \sigma(\overrightarrow{g})$. However, $\sigma(\overleftarrow{e}) = \overrightarrow{e} \leq \overrightarrow{g}$ but $\overleftarrow{e} \# \overleftarrow{h} \# \overrightarrow{g}$, contradicting the **interval condition**.

Do these conditions have any other consequences ?

Consider an anti-monotonic relation F, where xFa and xFb.

In any partial order $x \leq x$, so $a \leq b$ and $b \leq a$. Therefore $a = b$, and so F is **injective**.

Similarly, when the converse F^c is anti-monotonic, F is a **partial function**.

Corollary : For a transition relation $[w]$, inverse-monotonicity and undirectedness imply that $[w]$ is a **partial injective function**.

Questions :

1. Are these properties (inverse-monotonicity, the interval condition) preserved by Birget's Bizzare Composition ?
2. Can we obtain all planar diagrams (of partial injections) this way ?

Composing transition relations (I)

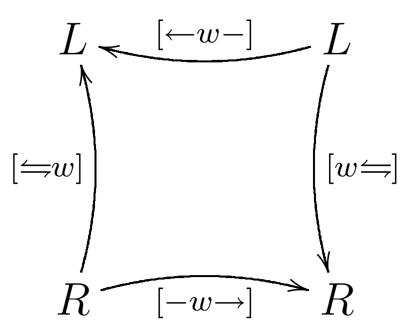
Each transition relation $[w] \subseteq Q \times Q$ may be *decomposed* into 4 components.

1. $[\leftarrow w -] \; \subseteq \; L \times L$
2. $[\leftrightharpoons w] \; \subseteq \; L \times R$
3. $[w \leftrightharpoons] \subseteq R \times L$
4. $[- w \rightarrow] \subseteq R \times R$.

This gives the **matrix** or **directed graph** of the transition relation

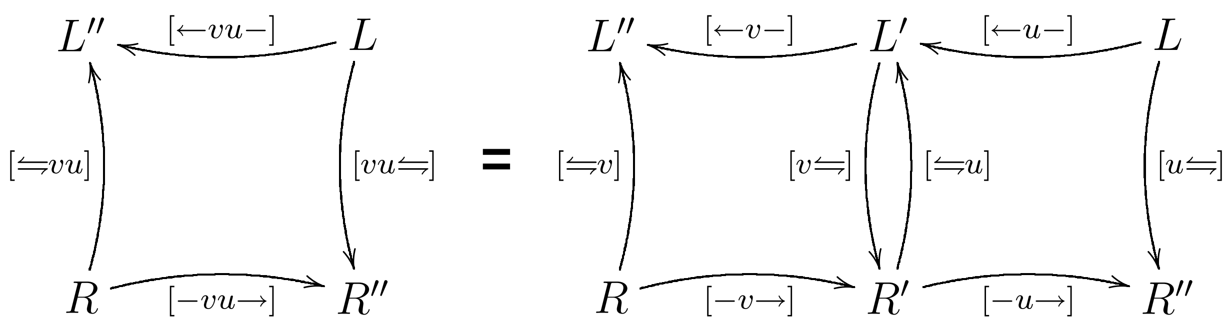
$$[w] = \begin{pmatrix} [\leftarrow w -] & [\rightleftharpoons w] \\ [w \rightleftharpoons] & [- w \rightarrow] \end{pmatrix}$$

drawn as :



Composing transition relations (II)

Given such graphs for $[v]$ and $[u]$, we draw the composite as



This concatenation denotes ‘taking the union over all paths”, giving

$$[\leftarrow vu-] = [\leftarrow v-] \bigcup_{n=0}^{\infty} ([\leftrightharpoons u][v \leftrightharpoons])^n [\leftarrow u-]$$

$$[\leftrightharpoons vu] = [\leftrightharpoons v] \ \cup \ [\leftarrow v-] \bigcup_{n=0}^{\infty} ([\leftrightharpoons u][v \leftrightharpoons])^n [-v \rightarrow]$$

and similarly (dually) for $[-vu \rightarrow]$ and $[vu \leftrightharpoons]$.

Composition and planarity

Using either

(i) Algebraic Manipulations, or

(ii) Categorical Structure (via the identification with Geometry of Interaction),

we may show :

1. This composition is **associative**
2. It also preserves **partial injectivity**
3. The composite of undirected transition relations is also **undirected**.
4. The composite of inverse-monotonic transition relations is also **inverse-monotonic**.
5. The composite of transition relations satisfying the **interval condition** also satisfies this condition.

There remain several worries –

Do our conditions capture *all* undirected planar diagrams ?

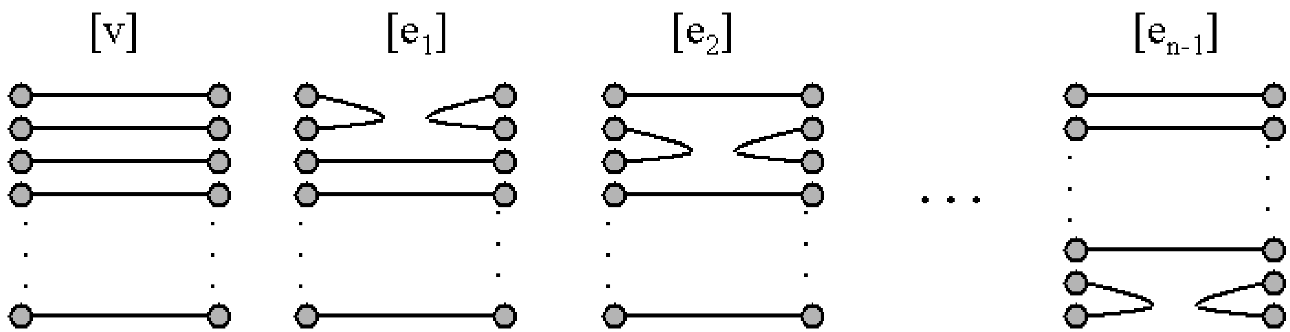
Do the conditions make everything trivial ?

Does composition correspond to the intuitive ideal of concatenating braids ?

Planar automata – an example

Define a two-way state-ordered automaton $\mathcal{TLA}_n$ to have

- State set $Q = L \uplus R$, where $L = \{\overleftarrow{1} \leq \overleftarrow{2} \leq \ldots \leq \overleftarrow{n}\}$ and $R = \{\overrightarrow{1} \geq \overrightarrow{2} \geq \ldots \geq \overrightarrow{n}\}$
- Input alphabet $A = \{v, e_1, e_2, \ldots, e_{n-1}\}$.
- Transition functions given by the following transition diagrams :



Properties of $\mathcal{T}LA_n$

It is easy to check that all transition functions :

1. are **undirected** (this is by construction!)
2. are **inverse-monotonic**.
3. satisfy the **interval condition**.

Using the GoI composition,

$$[e_i][e_j][e_i] = [e_i] \quad \text{when} \quad |i-j| = 1$$
$$[e_i][e_j] = [e_j][e_i] \quad \text{when} \quad |i-j| > 1$$
$$[e_i][e_i] = [e_i]$$

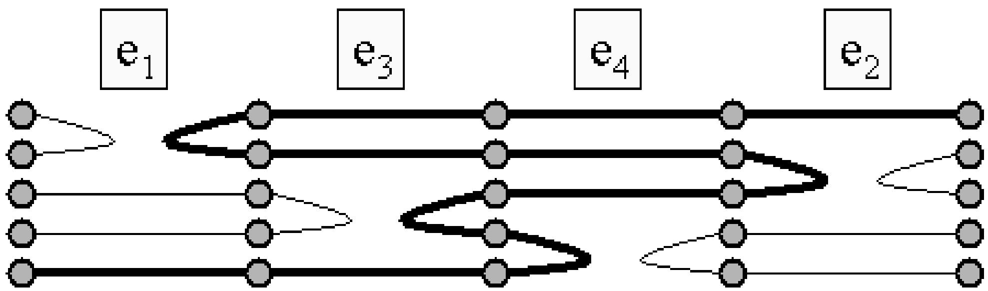
This gives a representation of the Temperley-Lieb monoid, with a loop value of 1.

More properties of $\mathcal{T}LA_n$

What is the **complexity** of these automata ?

— by complexity, we mean the max. number of steps between **boundary configurations**.

For $\mathcal{T}LA_5$, with 4 cells on the tape :

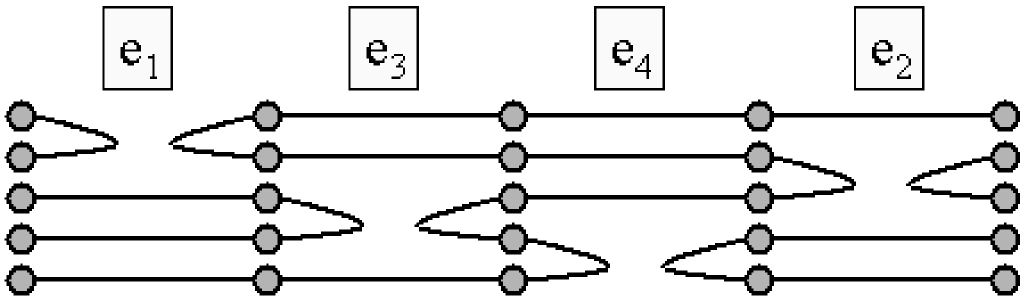


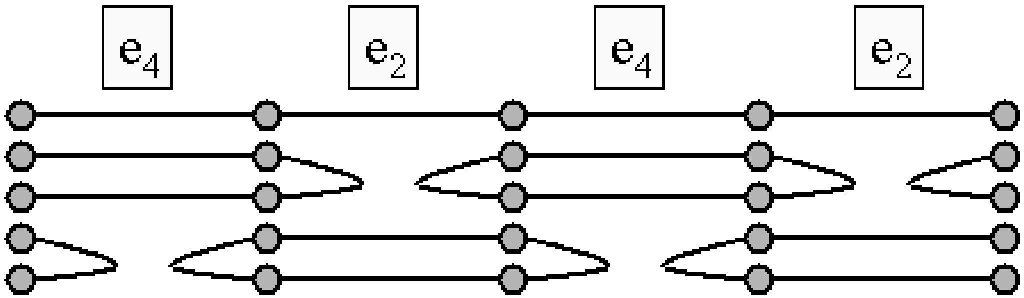
The longest path traverses 12 cells – i.e. takes 2 machine cycles before termination.

— a detailed analysis is easy, but uninteresting ...

A more interesting property of $\mathcal{T}LA_n$

Consider 2 distinct diagrams for $\mathcal{T}LA_5$, with 4 cells on the tape :





In both cases, the total length of all paths is 20$(= 5 \times 4)$ steps.

the general case :

A general result Given $\mathcal{T}LA_n$, with k cells on the tape,
the sum of all path lengths *including closed loops* is $n \times k$.

Question : Why is this useful ?

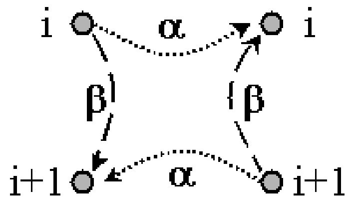
By counting the number of steps to termination of a two-way automaton, we can calculate the number of closed loops.

Taking ‘time-to-termination’ into account, we can represent the Temperley-Lieb monoid with a **non-trivial loop value**.

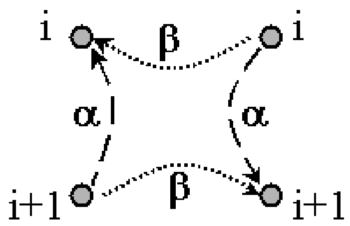
Future directions

- Modify the definition of quantum two-way automata, so the tape is not circular
  - This will allow for analogues of Birget's relations
  - ... but will involve 'splitting up computations by how long they take'.

- Consider transitions labelled by complex amplitudes, such as



and



  - Interpret this as 'a coherent superposition of left-moving and right-moving',
  - and require α and β to satisfy conditions related to the Jones polynomial.

Question : How much of the Jones polynomial algorithm is just a 2-way automaton computation ?